Linear Algebraic methods in Combinatorics

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A tale of four cities...

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Even City :

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3. No two clubs have the same set of members.

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Non-trivial fact : For any set of $\ell < 2^{20}$ clubs, we can add one more club.

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Question : Can there be more?

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They are linearly independent over $\mathbb{Z}_2$. (This is a strong statement.)
In a vector space of dimension 40, there cannot be more than 40 linearly independent vectors.

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Then take dot product with x_i to get $\lambda_i = 0$.
Do this for all i. Thus all λ_i's are zero. Hence the x_i's are linearly independent.

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Proof : Let $N = M^T M$.
Fact : For any field $\mathbb{F}$, $\text{rank}_{\mathbb{F}}(AB) \leq \min(\text{rank}_{\mathbb{F}}(A), \text{rank}_{\mathbb{F}}(B))$

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i.e. $t \leq \text{rank}_{\mathbb{Z}_2}(M) \leq \min(t, 40)$ and so $t \leq 40$.

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Question: Is the same true for Odd-even town?

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Reverse City : Again has 40 inhabitants.

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Question : How many clubs can there be? Think it over...

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If $S_1, S_2, \ldots, S_t$ are distinct subsets of $[n]$ such that for all $i \neq j$, $|S_i \cap S_j| = \ell$, then $t \leq n$.

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Let M be the $n \times t$ matrix and let $N = M^T M$.

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Let $K = \text{diag}[|S_i| - \ell]$.
It is easy to see that $N = \ell J + K$.

Ramsey Theory

Theorem 3

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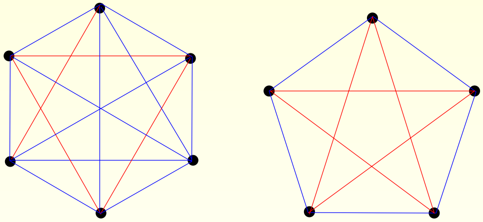

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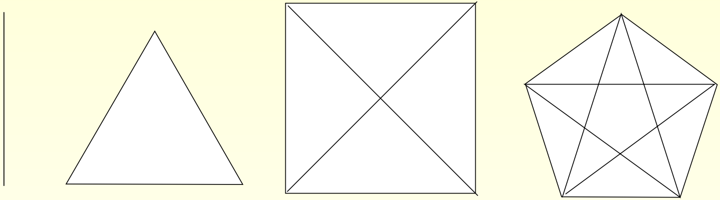

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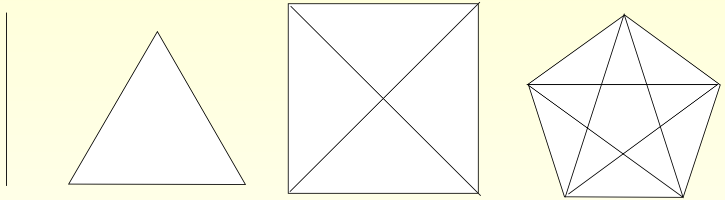

Call $R(rd, bl)$ as the minimum number of vertices such that *ANY* colouring of the edges of the complete graph on $R(rd, bl)$ vertices has a RED clique of size rd or a BLUE clique of size bl.

Ramsey Theory

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Theorem 4 (Ramsey, 1927)

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Theorem 4 (Ramsey, 1927)

For all p, q the number $R(p,q)$ is finite.

We don't know exact values of $R(p,q)$ for arbitrary p and q.

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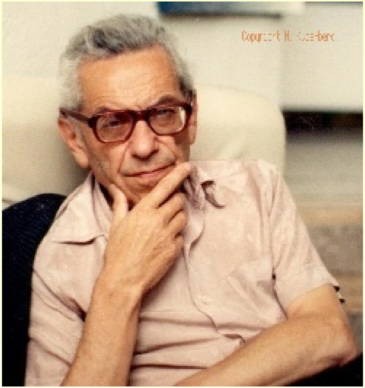

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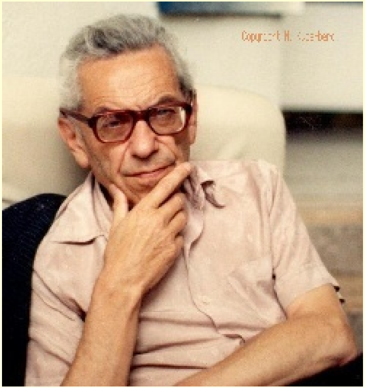

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Can we get bounds on these Ramsey Numbers?

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$(\sqrt{2})^k \leq R(k,k) \leq 4^k$. *Alas, the proofs of Erdös are probabilistic. We do not know an explicit family of graphs on* $(\sqrt{2})^k$ *vertices and a proof that the family is a Ramsey graph.*

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Nagy's theorem

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Argument against red cliques Red clique implies sets $S_1, S_2, \cdots, S_t$ such that for all $i \neq j$, $|S_i \cap S_j| = 0, 2$, but $|S_i| = 3$.

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Thus there is no monochromatic clique of size k!!

Graham-Pollak Theorem

Let K_n be the complete graph on n vertices. We want to cover all the edges of K_n using complete bipartite graphs K_{S_i, T_i} such that each edge of K_n occurs in precisely one K_{S_i, T_i} and use the minimum number of complete bipartite graphs.

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Theorem 8 (Graham-Pollak)

The minimum number of complete bipartite graphs needed to cover the edges of K_n (as a disjoint union) is $n-1$.

Proof: Assume that the vertex set of K_n is $\{1, 2, \ldots, n\}$. Associate a polynomial $P_G(x_1, \ldots x_n) = \sum_{e \in E(G), e=\{i,j\}} x_i x_j$ to the graph G.

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We clearly have $P_{K_n}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{q} P_{K_{S_i, T_i}}(x_1, x_2, \ldots, x_n)$.

It is easy to see that
$P_{K_n}(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right].$

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and that $P_{K_{S_a,T_a}}(x_1, x_2, \ldots, x_n) = (\sum_{i \in S_a} x_i)(\sum_{j \in T_a} x_j)$.

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Since $q \le n - 2$, this system has n variables and at most $n - 1$ equations.
Thus, over $\mathbb{R}$, there is a non-zero solution $(z_1, z_2, \ldots, z_n)$.

It is easy to see that
$P_{K_n}(x_1, x_2, \ldots, x_n) = \sum_{1 \le i < j \le n} x_i x_j = \frac{1}{2}\left[(\sum_{i=1}^{n} x_i)^2 - \sum_{i=1}^{n} x_i^2\right]$.
and that $P_{K_{S_a,T_a}}(x_1, x_2, \ldots, x_n) = (\sum_{i \in S_a} x_i)(\sum_{j \in T_a} x_j)$.
Consider the linear homogeneous system of equations $\sum_{i \in S_k} x_i = 0$ for $1 \le k \le q$ and $\sum_{i=1}^{n} x_i = 0$.
Since $q \le n-2$, this system has n variables and at most $n-1$ equations.
Thus, over $\mathbb{R}$, there is a non-zero solution $(z_1, z_2, \ldots, z_n)$.
This solution violates $P_{K_n}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{q} P_{K_{S_i,T_i}}(x_1, x_2, \ldots, x_n)$.
This gives us a contradiction.

The end...

Questions/ Comments???