On a degenerate algebraic Riccati equation

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A linear ODE

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where u_0 and $\mathrm{u}(t)$ are in $\mathbb{R}^{\mathbb{N}}$ for each $t \in \mathbb{R}$.
We can now write down the solution to this equation in the form

$$\mathrm{u}(t) = e^{tA}\mathrm{u}_0, \; t \in \mathbb{R},$$

where, for any $N \times N$ matrix B, the matrix e^B is defined by the usual exponential series, *i.e.*

$$e^B = I + \sum_{n=1}^{\infty} \frac{B^n}{n!},$$

Examples

Example If A is a diagonal matrix $\mathrm{diag}\{d_1, \cdots, d_N\}$, then

$$e^{tA} = \mathrm{diag}(e^{td_1}, \cdots, e^{td_N}). \blacksquare$$

Example Let $\alpha, \omega \in \mathbb{R}$.

$$A = \left[\begin{array}{rr} \alpha & \omega \\ -\omega & \alpha \end{array}\right].$$

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$$e^{tA} \;=\; e^{\alpha t}\left[\begin{array}{cc} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{array}\right]. \;\blacksquare$$

Now, if V is any Banach space and $A \in \mathcal{L}(V)$, we can till define e^A via the exponential series and $e^A \in \mathcal{L}(V)$. Now if we have the initial value problem

$$\begin{array}{rcl} \frac{du}{dt}(t) & = & Au(t), \quad t \in \mathbb{R}, \\ u(0) & = & u_0, \end{array}$$

where u_0 and $u(t)$ are in V for all $t \in \mathbb{R}$, we still can write the solution as

$$u(t) \;=\; e^{tA}u_0, \;\; t \in \mathbb{R}.$$

Partial differential equations

Many linear partial differential equations of evolution type can be put in the following format: Let V be a Banach space (often a Hilbert space). Let $u_0 \in V$. Find $u(t) \in V$ such that the mapping $t \mapsto u(t)$ is differentiable on $\{t > 0\}$ and

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Example

Let $V = \mathcal{C}[0,1]$. Define $D(A) = \mathcal{C}^1[0,1]$ and, for $u \in D(A)$, define Au by

$$(Au)(t) \ = \ u'(t), \ t \in [0,1].$$

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Since the initial value problem is linear and we expect the solution to be continuous with repect to the data, the mapping $u_0 \mapsto u(t)$ must be continuous and linear for each t. Let us set $u(t) = S(t)u_0$. Thus, for each $t > 0, S(t) \in \mathcal{L}(V)$.

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$$S(t_1)(S(t_2)u_0) \ = \ S(t_1 + t_2)u_0.$$

This leads us to the following definition:

C_0- Semigroups

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for every $u \in V$. ■

Remark The second condition is the semigroup property. The continuity assumption in (iii) gives rise to C_0 in the name. ■

Examples

Example If $A \in \mathcal{L}(V)$, then $S(t) = e^{tA}$ defines a semigroup for $t \geq 0$. In fact, $S(t)$ is defined for all $t \in \mathbb{R}$ and forms a group. ■

Example (**Translation semigroup**) Let V be the space of bounded and uniformly continuous real-valued functions defined on the real line. For $f \in V$, define

$$S(t)f(s) \ = \ f(t+s), \ s \in \mathbb{R}.$$

Again $\{S(t)\}_{t \geq 0}$ defines a semigroup. In this case also we have a group since $S(t)$ can be defined for all $t \in \mathbb{R}$. ■

Infinitesimal generator

Given a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space V, we define an unbounded operator, A, called the **infinitesimal generator** of the semigroup as follows:

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Example If V is the space of bounded uniformly continuous real-valued functions defined on $\mathbb{R}$, then the domain of the infinitesimal generator of the translation semigroup defined earlier is the subspace of continuously differentiable functions in V and such that the derivative is also in V; for such functions u, $Au = u'$. ■

Infinitesimal generator, contd.

Theorem

Let V be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup defined on V. Let A be the infinitesimal generator of this semigroup. Then, if $u_0 \in D(A)$, the mapping $t \mapsto S(t)u_0$ is continuously differentiable. Further

$$\frac{d(S(t)u_0)}{dt}(t) \;=\; AS(t)u_0 \;=\; S(t)Au_0. \blacksquare$$

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$$\frac{d(S(t)u_0)}{dt}(t) \;=\; AS(t)u_0 \;=\; S(t)Au_0. \;\blacksquare$$

Thus, if the initial value u_0 is in $D(A)$, then the initial value problem (1) has a (unique) solution $u(t) = S(t)u_0$.

Consider the translation semigroup and its infinitesimal generator. Let u_0 be a continuously differentiable, uniformly continuous and bounded real-valued function defined on $\mathbb{R}$, *i.e.* $u_0 \in D(A)$, where A is the infinitesimal generator. For $u \in D(A)$, recall that $Au = u'$. Thus the initial value problem (1) reads as

$$\frac{\partial u}{\partial t}(t,x) \; = \; \frac{\partial u}{\partial x}(t,x), \text{ for } t > 0, \; u(0,x) \; = \; u_0(x).$$

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Then the solution is given by $u(t,x) = u_0(t+x)$. This is called the **transport equation**. ■

Given an initial value problem (1), the question, therefore, is whether A is the infinitesimal generator of a C_0-semigroup. This is answered by the **Hille-Yosida theorem**, which characterises such operators. The infinitesimal generator of a C_0-semigroup will always be closed and densely defined with additional spectral properties.

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A special case of the above theorem in Hilbert spaces is particularly useful. Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be an unbounded operator. It is said to be **dissipative** if $(Au, u) \leq 0$ for every $u \in D(A)$. In addition, if the range of $I - A = H$, *i.e.* for every $y \in H$, there exists $u \in D(A)$ such that $u - Au = y$, it is said to be **maximal dissipative**. We can show that maximal dissipative operators are the infinitesimal generators of **contraction semigroups**, *i.e.* the semigroup $\{S(t)\}_{t \geq 0}$ generated by A is such that $\|S(t)\| \leq 1$ for every $t \geq 0$.

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Control Problems

Let Z be a Hilbert space and let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on Z. Let U be a Hilbert space called the space of **controls** and let $B : U \to Z$. Let $\zeta \in Z$. Consider the problem:

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The space Z is called the **state space** and the solution of (2), $z \in L^2(0, T; Z)$, is the **state** of the system. The operator B is called the **control operator** and u is the **control** applied to influence the evolution of the solution of (2). We may have several objectives from the point of view of control of the above system.

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For simplicity and uniformity of notation, we will denote the semigroup generated by A by $\{e^{tA}\}_{t\geq 0}$.

Controllability

- **Exact controllability**
  The system (2) is **exactly controllable** in time $T > 0$, if for any initial data ζ, and any given element ζ_1, there exists a control $u \in L^2(0, T; U)$ such that $z(T) = \zeta_1$.

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- **Null controllability**
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  For every initial value ζ, there exists a control u such that $z(T) = 0$. This happens if, and only if, for all $\zeta \in Z$,

$$\int_0^T \|B^* e^{tA^*}\zeta\|_Z^2 \, dt \ \geq \ C\|e^{TA^*}\zeta\|_Z^2.$$

All three notions are equivalent if the spaces are finite dimensional. This is not true in infinite dimensions. Of course exact controllability, trivially, implies the other two.

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The heat equation is an example of an approximately controllable system. It cannot be exactly controllable because, given any initial data, however rough, the solution instantaneously becomes smooth (analytic). Unlike the wave equation, which can be solved backwards (the semigroup in this case is a group), the heat equation cannot be solved backwards.

Optimal Control: Linear Regulator Problem

Let Y be a Hilbert space, called the space of **observation**. Let $C : Z \to Y$ be a bounded linear operator. Let $\zeta \in Z$ be fixed. Let $z(t)$ be the state, *i.e.* the solution of (2). Let $y_d \in Y$ be given. It may be, say, the observation which one 'desires' to make. We then try to come as close to this by penalization, in the least squares sense. In other words, we wish to minimize $\int_0^T \|Cz(t) - y_d\|_Y^2 \, dt$. However, to arrive at $z(t)$, we have exercised a control u which contributes to the cost.

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$$J(z,u) \;=\; \frac{1}{2}\int_0^T \|Cz(t) - y_d\|_Y^2 \, dt + \frac{1}{2}\int_0^T \|u(t)\|_U^2 \, dt.$$

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Problem: Find $u \in L^2(0,T;U)$ such that J is minimized.
This is called an optimal control problem with **finite** time horizon, and the optimal solution u is called the **optimal control**.

Optimal Control: infinite time horizon

Let $\zeta \in Z$ be fixed. Let $z(t)$ be the state, *i.e.* the solution of (2). Define the cost functional

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Finite Cost Condition

For every $\zeta \in Z$, there exists a control $u \in L^2(0,\infty;U)$ such that $J(z,u) < \infty$.

If FCC holds then there exists a unique optimal control which minimizes J.

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$$P \;=\; P^* \;\geq 0$$

and P satisfies the **algebraic Riccati equation**

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Then (2) becomes

$$z'(t) \ = \ (A - BB^*P)z(t), \ t > 0, \ \text{and} \ z(0) \ = \ \zeta.$$

In the case of a finite time horizon, defined by T, we have a **differential** Riccati equation. Given the system (2), we have the dual system:

$$\begin{array}{rcl} -p'(t) & = & A^*p(t) + C^*Cz, \; 0 < t < T, \\ p(T) & = & C^*Cz(T). \end{array}$$

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- The pair (A, C) is said to be **exponentially detectable** if there exists $L \in \mathcal{L}(Y, Z)$ such that the operator $A + LC$, with domain $D(A)$ is exponentially stable.
- If A is exponentially stable, then FCC automatically holds.
- If FCC holds and the pair (A, C) is exponentially detectable, then the solution to the algebraic Riccati equation is unique.

Numerical Approximation

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Benner: Choose P_0 such that $P_0 = P_0^* \geq 0$ solution to the **degenerate** algebraic Riccati equation:

$$A_h^* P + P A_h - P B_h B_h^* P \ = \ 0$$

which is also such that $A_h - B_h B_h^* P_0$ is exponentially stable.

We are interested in the following problem:
Find $P \in \mathcal{L}(Z)$ such that $P = P^* \geq 0$ such that

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and such that $A - BB^*P$ is exponentially stable.
Let us assume for the time being that the spaces Z and U and Y are **finite dimensional**.

A Comparison Principle

Lemma

Let $C_i \in \mathcal{L}(Z, Y)$ *for* $i = 1, 2$ *be such that* $C_1^* C_1 \geq C_2^* C_2$ *Let* $P_i \in \mathcal{L}(Z)$ *be such that* $P_i = P_i^* \geq 0$ *and*

$$A^* P_i + P_i A - P_i B B^* P_i + C_i^* C_i \; = \; 0$$

for $i = 1, 2$. *If* $A - BB^* P_1$ *is exponentially stable, then* $P_1 \geq P_2$ ■

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for $i = 1, 2$. *If* $A - BB^* P_1$ *is exponentially stable, then* $P_1 \geq P_2$ ■

Corollary

The algebraic and degenerate algebraic Riccati equations admit at most one solution P *such that* $A - BB^* P$ *is exponentially stable. In particular, if* A *is itself exponentially stable, then the degenerate equation has no non-trivial solutions such that* $A - BB^* P$ *is exponentially stable.* ■

A Special Case

Theorem

The following are equivalent:
(i) The operator $-A$ *is exponentially stable and there exists* $\alpha > 0$ *such that*

$$\int_0^\infty \|B^* e^{-tA^*} z\|_Z^2 \, dt \ \geq \ \alpha \|z\|_Z^2 \tag{3}$$

for all $z \in Z$.

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(ii) The degenerate algebraic Riccati equation admits solution $P \in \mathcal{L}(Z)$ which is **invertible** *and such that $A - BB^*P$ is exponentially stable.*

Proof: Step 1: If $-A$ is exponentially stable and (3) holds,

$$Q \;=\; \int_0^\infty e^{-tA} BB^* e^{-tA^*} \, dt$$

is well defined and $Q = Q^*$. Further, for any $z \in Z$, (3) implies that

$$(Qz, z)_Z \;\geq\; \alpha \|z\|_Z^2.$$

Thus, Q is invertible, and $Q > 0$.

Step 2. Let $Q(t) = e^{-tA}BB^*e^{-tA^*}$. Then

$$BB^* = Q(0) = -\int_0^\infty \frac{d}{dt}Q(t)\, dt$$

and so we deduce that

$$AQ + QA^* = BB^*.$$

Set $P = Q^{-1}$.
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Set $P = Q^{-1}$.
Multiplying both sides of the equationby P, we get that P satisfies the degenerate equation.
Step 3. Since P is invertible, and solves the degenerate equation, we see that

$$P(A - BB^*P)P^{-1} \; = \; -A^*$$

and the RHS is, by assumption, also exponentially stable. Thus $A - BB^*P$ is also exponentially stable.
Step 4. The converse is proved by essentially retracing this proof. ■

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Remark 2. If the pair $(-A, B)$ is exactly controllable in some time $T > 0$, then there exists $\alpha > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} z\|_Z^2 \, dt \; \geq \; \alpha \|z\|_Z^2$$

for all $z \in Z$ and so (3) is also true. Thus, the above theorem is applicable if $-A$ is exponentially stable and the pair $(-A, B)$ is exactly controllable.

The General Case

Henceforth, we will assume the following to hold:
(H) There exist subspaces Z_s and Z_u of Z such that
(i) $Z = Z_s \oplus Z_u$.
(ii) Z_s and Z_u are invariant under A.
(iii) The restriction of A to Z_s is exponentially stable.
(iv) The restriction of $-A$ to Z_u is exponentially stable.

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(iv) The restriction of $-A$ to Z_u is exponentially stable.

Example The matrix A has no eigenvalues on the imaginary axis. Then we can find invariant subspaces Z_s and Z_u such that all the eigenvalues of the restriction of A to Z_s are with negative real part and all the eigenvalues of the restriction of A to Z_u have positive real part. ■

Let $\pi_s : Z \to Z_s$ and $\pi_u : Z \to Z_u$ be the canonical projections with respect to this decomposition of Z.

$$\pi_u A = A\pi_u = \pi_u A \pi_u.$$

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Assume that the pair $(\pi_u A, \pi_u B)$ is such that there exists $\alpha > 0$ satisfying:

$$\int_0^\infty \|(\pi_u B)^* e^{-t(\pi_u A)^*} z\|_Z^2 \, dt \ \geq \ \alpha \|z\|_Z^2 \tag{4}$$

for all $z \in Z_u$.

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for all $z \in Z_u$. Then, by Theorem 1, there exists $P_u \in \mathcal{L}(Z_u)$ such that $P_u = P_u^* \geq 0$ and

$$P_u(\pi_u A) + (\pi_u)^* P_u - P_u(\pi_u B)(\pi_u B)^* P_u \ = \ 0.$$

Further, $\pi_u A - (\pi_u B)(\pi_u B)^* P_u$ is exponentially stable.

Theorem

Assume that the hypothesis (H) holds and that (4) is true. Let $P_u \in \mathcal{L}(Z_u)$ *be as detailed earlier. Define*

$$P \;=\; \pi_u^* P_u \pi_u.$$

Then $P = P^* \geq 0$; P *satisfies the degenerate algebraic Riccati equation and* $A - BB^*P$ *is exponentially stable.*

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Then $P = P^* \geq 0$; P *satisfies the degenerate algebraic Riccati equation and* $A - BB^*P$ *is exponentially stable.*

Proof: Clearly P is self-adjoint and non-negative. That it satisfies the degenerate algebraic Riccati equation follows by multiplying the equation for P_u on the left by π_u^* and on the right by π_u and using the fact that π_u commutes with A (and so its adjoint commutes with A^*) and that π_u is a projection.

Finally we see that (with respect to the decomposition $Z = Z_s \oplus Z_u$),

$$\begin{bmatrix} \pi_s(A - BB^*P)z \\ \pi_u(A - BB^*P)z \end{bmatrix} =$$

$$= \begin{bmatrix} \pi_s A & -\pi_s BB^*\pi_u^* P_u \\ 0 & \pi_u A - \pi_u BB^*\pi_u^* P_u \end{bmatrix} \begin{bmatrix} \pi_s z \\ \pi_u z \end{bmatrix}.$$

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Since both diagonal blocks of the upper triangular matrix are exponentially stable, it follows that $A - BB^*P$ is also exponentially stable. ■

Assume that A has no eigenvalues on the imaginary axis.
If $(-A, B)$ is exactly controllable in time $T > 0$, then so is $(-\pi_u A, \pi_u B)$.
The eigenvalues of $\pi_s A$ are precisely those of A with negative real part.
Since $\pi_u A - \pi_u BB^* \pi_u^* P_u$ is similar to $-(\pi_u A)^*$, the eigenvalues of this matrix are the reflections on the imaginary axis of those of A with positive real part.
Thus, the eigenvalues of $A - BB^* P$ are those of A with negative real part and the reflections on the imaginary axis of those eigenvalues of A with positive real part.

A Variational Characterization

Back to the special case:
We will assume that $-A$ is exponentially stable and that (3) holds. Let P be the solution to the degenerate algebraic Riccati equation obtained in the proof of Theorem 1.

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Lemma

Let H be a real Hilbert space. Let $\{G_n\}$ be a sequence in $\mathcal{L}(H)$ such that $G_n = G_n^* \geq 0$. Assume, further that, for every $v \in H$, the sequence $\{(G_n v, v)_H\}$ is decreasing. Then, there exists $G \in \mathcal{L}(H)$ such that $G = G^* \geq 0$ and, for every $v \in H$, $G_n v \to Gv$ in H. ■

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Since $A - BB^*P$ is exponentially stable, (A, I) is exponentially detectable. So is the pair (A, kI) for any $k \in \mathbb{R}$. In particular, for every $\varepsilon > 0$, there exists a unique $P_\varepsilon = P_\varepsilon^* \geq 0$ such that

$$P_\varepsilon A + A^* P_\varepsilon - P_\varepsilon BB^* P_\varepsilon + \varepsilon^2 I \ = \ 0.$$

Further, $A - BB^*P_\varepsilon$ is exponentially stable.
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Since $A - BB^*P$ is exponentially stable, we get, by the comparison principle, that $P \geq P_0$. Again, by the same principle, we have $P_\varepsilon \geq P$ and, passing to the limit, $P_0 \geq P$. Thus, $P_0 = P$.

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Let $\zeta \in Z$ be fixed such that $\zeta \neq 0$.
Now, for $u \in L^2(0, \infty; U)$, set z_u to be the solution of (2). Define

$$E_\zeta \ = \ \{u \in L^2(0, \infty; U) \mid z_u \in L^2(0, \infty; Z\}.$$

Consider

$$\min_{u \in E_\zeta} \int_0^\infty \|u(t)\|_U^2 \, dt.$$

FCC $\Rightarrow E_\zeta \neq \emptyset$.
E_ζ closed?

Proposition

If $-A$ is exponentially stable and (3) holds, then the above minimization problem admits a solution. We have

$$(P\zeta, \zeta)_Z \;=\; \min_{u \in E_\zeta} \int_0^\infty \|u(t)\|_U^2 \, dt$$

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Remark 3. Since $-A$ is exponentially stable, A is NOT and so $0 \notin E_\zeta$.

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Thus,

$$a - b^2p \;=\; \begin{cases} a & \text{when } a < 0 \\ -a & \text{when } a > 0. \end{cases}$$

Thank You!