Introduction to Topology
A Short Course

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Preface

These lecture notes are based on class materials prepared for students to supplement the course *Introduction to Topology*, which I taught at BITS Pilani, K.K. Birla Goa Campus, during the August - December 2023 semester. The material has since been refined by incorporating numerous illustrative examples and detailed remarks.

While the topics covered are standard - similar to those found in popular textbooks like *Topology: A First Course* by J. Munkres or *Introduction to General Topology* by K.D. Joshi - the primary distinction lies in the sequencing. By framing the subject as a natural evolution of concepts from Real Analysis, students can explore foundational topological notions within a more general framework that requires less initial structure. Essential set-theoretic concepts are provided in the Appendix for ready reference.

Designed strictly for a one-semester course, these notes exclude advanced topics such as “metrization theorems” and “homotopy theory”. However, the foundational background provided here will fully equip students to pursue those advanced subjects independently using the reference texts mentioned above.

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Note to the Reader

Set-theoretic notations. Throughout the book, we shall use standard set-theoretic notations such as

$$\cup, \ \cap, \ \subseteq, \ \subset, \ \in$$

to denote 'union', 'intersection', 'subset of', 'proper subset of', 'belong(s) to', respectively. For sets S_1 and S_2, the set $\{x \in S_1 : x \notin S_2\}$ is denoted by $S_1 \setminus S_2$. If f is a function with domain S_1 and codomain S_2, then we use the notation $f : S_1 \to S_2$, and it is also called a 'map' from S_1 to S_2. For more on set-theoretic notions, one may see the *Appendix*.

Also, we use the standard notations such as

$$\begin{aligned}
\mathbb{N} &: \text{ set of all positive integers} \\
\mathbb{Z} &: \text{ set of all integers} \\
\mathbb{Q} &: \text{ set of all rational numbers} \\
\mathbb{R} &: \text{ set of all real numbers} \\
\mathbb{C} &: \text{ set of all complex numbers} \\
:= &: \text{ is defined by} \\
\forall &: \text{ for all} \\
\exists &: \text{ there exists or there exist} \\
\Longrightarrow &: \text{ implies or imply} \\
\Longleftrightarrow &: \text{ if and only if} \\
\mapsto &: \text{ maps to}
\end{aligned}$$

Attention! It is a well-known dictum that "mathematics cannot be learned without paper and pencil in hand." Of course, you may replace paper and pencil with any corresponding modern gadget. The crux of the matter is that you must work out the details yourself while learning mathematics.Both teachers and students are urged to draw appropriate figures to represent abstract ideas geometrically using specific examples.

Examples and exercises are integral parts of this course. Many of the examples are presented as statements that the reader must verify. Additionally, many of the problems in Chapter 9 repeat examples and exercises from the text; they are isolated at the end solely to reinforce the study material. Readers are strongly urged to solve all problems in detail.

Contents

Chapter 1

Topology: Motivation, Definitions and Examples

In this chapter, we motivate the subject of *topology* through the notion of open sets already known in the context of metric spaces, and provide many of the basic definitions and illustrative examples.

1.1 Motivation

Many of the notions introduced in real analysis, in the setting of metric spaces, can be expressed using the notion of open sets without the explicit reference to the metric.

Let X be a metric space space and let $\mathcal{G}$ be the collection of all open sets in X. Recall that, by an *open set in a metric space* X we mean a set $A \subseteq X$ such that for every $x \in X$, there exists an open ball centered at x and contained in X. Recall that an *open ball* centered at $x \in A$ is a set of the form

$$B_d(x, r) := \{y \in X : d(x, y) < r\}$$

for some $r > 0$, where d is the metric on X. (The open ball $B_d(x, r)$is also denoted by $B(x, r)$ if the metric is understood from the context.)

Now, let us look at the following characterizations:

(1) A sequence (x_n) in X *converges* to $x \in X$ iff for every $A \in \mathcal{G}$ with $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$.

(2) If $S \subseteq X$ and $x \in X$, then $x \in X$ is a *closure point* of S iff every $A \in \mathcal{G}$ which contains x contains at least one element of S.

(3) If $S \subseteq X$ and $x \in X$, then x is a *limit point* (accumulation point) of S iff every $A \in \mathcal{G}$ which contains x contains at least one element of S other than x.

(4) If Y is also a metric space (possibly different from X), then a function

> $f : X \to Y$ is *continuous* at a point $x \in X$ iff for every open set V in Y containing $f(x)$, there exists an open U in X containing x such that $f(U) \subseteq V$.

To warm up your memory, let us give proofs for the above four statements.

Proof of (1). Let (x_n) be a sequence in X which converges to $x \in X$. Let A be an open set containing x. Then, by the definition of an open set, there exists $r > 0$ such that $B_d(x, r) \subseteq A$, and by the definition of the convergence, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < r$ for all $n \geq n_0$. Hence, $x_n \in B_d(x, r)$ for all $n \geq n_0$. Since $B_d(x, r) \subseteq A$, we obtain $x_n \in A$ for all $n \geq n_0$.

Conversely, suppose that for every open set A containing x there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. We prove that $d(x_n, x) \to 0$ as $n \to \infty$. For this, let $\varepsilon > 0$ be an arbitrary number. Since $B_d(x, \varepsilon)$ is an open set, by hypothesis, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_d(x, \varepsilon)$ for all $n \geq n_0$. That is, $d(x_n, x) < \varepsilon$ for all $n \geq N$. This means that $d(x_n, x) \to 0$ as $n \to \infty$. □

Proof of (2). Let $S \subseteq X$ and $x \in X$. Suppose x is a closure point of S. Then, for every open ball centered at x contains at least one point from S. Now, let A be any open set containing x. Since A is open there exists an open ball $B_d(x, r) \subseteq A$. Since $B_d(x, r)$ contains at least one point from S, A also contains at least one point from S.

Conversely, suppose that every open set A which contains x contains at least one element of S. Then, in particular, every open ball centered at x contains one element of S. □

Proof of (3). Proof of (3) is similar to the proof of (2). □

Proof of (4). Let Y be a metric space with a metric ρ and $x \in X$. Suppose $f : X \to Y$ be continuous at x. Now, let V be an open set in Y containing $f(x)$. Then there exists an $\varepsilon > 0$ such that $B_\rho(f(x), \varepsilon) \subseteq V$. Since f is continuous at x, there exists Then, we know that for every $x \in X$, there exists $\delta > 0$ such that

$$d(x, u) < \delta \quad \Longrightarrow \quad \rho(f(x), f(u)) < \varepsilon,$$

that is

$$u \in B_d(x, \delta) \quad \Longrightarrow \quad f(u) \in B_\rho(f(x), \varepsilon).$$

Thus, taking $U = B_d(x, \delta)$, we obtain $f(U) \subseteq B_\rho(f(x), \varepsilon) \subseteq V$.

Conversely, suppose that for every open set V in Y containing $f(x)$, there exists an open U in X containing x such that $f(U) \subseteq V$. Now, let $\varepsilon > 0$ be given, and let $V = B_\rho(f(x), \varepsilon)$. Let U be an open set in X containing x

such that $f(U) \subseteq V$. Then there exists $\delta > 0$ such that $B_d(x, \delta) \subseteq U$. Since $f(U) \subseteq V$, we see that $u \in U$ implies $f(u) \in V$. In particular, $u \in B_d(x, \delta)$ implies $f(u) \in B_\rho(f(x), \varepsilon)$, that is,

$$d(x, u) < \delta \quad \Longrightarrow \quad \rho(f(x), f(u)) < \varepsilon.$$

This shows that f is continuous at x. $\square$

We shall see that the following three properties of $\mathcal{G}$ are sufficient to study many of the notions in metric space theory, and if we build a theory based on these properties, then it also include many instances beyond the setting of metric spaces as well.

1. $\varnothing \in \mathcal{G}$ and $\Omega \in \mathcal{G}$;
2. Union of any arbitrary collection of members of $\mathcal{G}$ is a member of $\mathcal{G}$;
3. Intersection of any two members of $\mathcal{G}$ is a member of $\mathcal{G}$.

Motivated by the above discussion, we introduce the notion of a *collection of open sets* in an arbitrary set X, which is known as a *topology*[1] on X, which essentially has only the above listed three properties.

1.2 Definitions and examples

Definition 1.1 (Topology) Let X be a set. Then a collection $\mathcal{T}$ of subsets of X is called a **topology** on X if it has the following properties:

1. $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
2. Union of any arbitrary collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$;
3. Intersection of any two members of $\mathcal{T}$ is a member of $\mathcal{T}$.

A set X together with a topology $\mathcal{T}$ on it is called a **topological space**, and in that case, the pair $(X, \mathcal{T})$ is also called a **topological space**. $\Diamond$

[1] The creater of the subject *Topology* can be considered as Riemann in the middle of nineteenth century, while he was working on theory of functions and their geometric properties. The term topology was introduced by J.B.Listing (a student of Gauss) in 1847, but the term became common only after Fréchet used it in his thesis. The subject was called *Analysis situs* earlier.

Definition 1.2 Let $(X, \mathcal{T})$ be a topological space. Then, members of $\mathcal{T}$ are called **open sets** in X, and a subset of X is said to be a **closed set** if its complement in X is an open set in X. $\Diamond$

Exercise 1.3 Let $(X, \mathcal{T})$ be a topological space. Show that, if $A_i \in \mathcal{T}$ for $i = 1, \ldots n$ for some $n \in \mathbb{N}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$. Deduce that finite union of closed sets is closed. $\lhd$

- If a subset of a topological space is not open, then it need not be closed, and a set which is not closed need not be open.

Example 1.4 Let X be a nonempty set with atleast two elements, and let A be a nonempty proper subset of X. Then

$$\mathcal{T} = \{\varnothing, A, X\}$$

is a topology on X. Note that A is open, but not closed, and $A^c := X \setminus A$ is closed but not open. $\Diamond$

Example 1.5 Let X be an arbitrary set and let $\mathcal{T} = \{\varnothing, X\}$. We note that $\mathcal{T}$ is a topology on X. If X contains at least two elements, then we see that a nonempty proper subset of X is neither open nor closed. $\Diamond$

Definition 1.6 For any set X, the family $\mathcal{T} = \{\varnothing, X\}$ of subsets of X is called the **indiscrete topology** on X. $\Diamond$

In contrast to Example 1.4 and Example 1.5, we have the following example, in which every subset is open as well as closed.

Example 1.7 Let X be an arbitrary set, and let $\mathcal{T} = 2^X$, the power set of X, namely, the family of all subsets of X. Then we note that $\mathcal{T}$ is a topology on X, and every subset of X is open and closed. $\Diamond$

Definition 1.8 For a set X, the topology $\mathcal{T} = 2^X$, the power set of X, namely, the family of all subsets of X, is called the **discrete topology** on X. $\Diamond$

The next example is one of the most important class of examples of a topological space.

Example 1.9 Let X be a metric space. We have already observed that the collection of all open subsets of a metric space satisfies the requirements for a topology. Thus, every metric space is a topological space with the topology consisting of all open subsets of X. $\Diamond$

Definition 1.10 (Metrizable space) A topological space $(X, \mathcal{T})$ is called **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is nothing but the set of all open sets with respect to the metric d, and this topology is often denoted by $\mathcal{T}_d$. $\Diamond$

Exercise 1.11 Let (X, d) be a metric space. Show that a subset A of X is open iff A is a union of open balls in X. $\lhd$

Exercise 1.12 Show that the a topology on a nonempty set X is the discrete topology iff every singleton subset of X is open and, in that case, the topology is induced by the discrete metric on X, namely, the metric d defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \qquad \lhd$$

Exercise 1.13 Show that the indiscrete topology on a nonempty set X is induced by a metric iff X is singleton, and in that case this topology is the discrete topology. $\lhd$

Recall that if X is a subset of $\mathbb{R}^k$ or $\mathbb{C}^k$, then d defined by

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$$

for $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in X is a metric on X, and it called the **usual metric** on X. Note that for $k = 1$,

$$d(x, y) = |x - y|.$$

Definition 1.14 On a subset of $\mathbb{R}^k$ (respectively, $\mathbb{C}^k$), the topology induced by the usual metric is called the **usual topology** on $\mathbb{R}^k$ (respectively, $\mathbb{C}^k$). $\Diamond$

- Every finite subset of $\mathbb{R}^k$ (respectively, $\mathbb{C}^k$) is closed with respect to the usual topology.

Remark 1.15 We have already noted that a set which is not open need not be closed; a set which is not closed need not be open; a set can be both open and closed. For instance, let $\mathbb{R}$ be with the usual topology. Then $[0, 1)$ is neither open nor closed, and the sets $\varnothing$ and $\mathbb{R}$ are both open and closed. In fact, with respect to the usual topology on $\mathbb{R}$, there is no other set which is both open and closed. Did you observe this in your real analysis class? $\Diamond$

We may also recall that two metrics, d and ρ, on a set X are said to be **equivalent metrics**, if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y) \quad \forall\, x, y \in X.$$

Theorem 1.16 *The topologies on set X induced by any two equivalent metrics are the same.*

Proof. Suppose d and ρ are equivalent metrics on X. We have to show that $\mathcal{T}_d = \mathcal{T}_\rho$. So, let $A \in \mathcal{T}_d$. Let $x \in A$. Then there exists $r > 0$ such that $B_d(x, r) \subseteq A$; that is,

$$y \in X,\ d(x,y) < r \quad \Longrightarrow \quad y \in A.$$

Note that, for $y \in X$,

$$\rho(x,y) < c_1 r \quad \Longrightarrow \quad d(x,y) < r.$$

Thus, we obtain

$$B_\rho(x, c_1 r) \subseteq B_d(x, r).$$

In particular, $B_\rho(x, c_1 r) \subseteq A$. This shows that $A \in \mathcal{T}_\rho$. Thus, we have proved that $\mathcal{T}_d \subseteq \mathcal{T}_\rho$. Analogously, it can be shown that $\mathcal{T}_\rho \subseteq \mathcal{T}_d$. $\square$

Exercise 1.17 Consider the metrics d_1, d_2, d_∞ on any subset X of $\mathbb{R}^2$ defined by

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + |x_2 - y_2|, \\ d_2(x,y) &= (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2}, \\ d_\infty(x,y) &= \max\{|x_1 - y_1|, |x_2 - y_2|\}, \end{aligned}$$

respectivlely, for $x, y \in X$. Show that the topologies induced by d_1, d_2, d_∞ are the same. $\lhd$

Exercise 1.18 Let X be a nonempty set and let d and ρ be metrics on X. Suppose that for every $x \in X$ and for every $r > 0$, there exists $s > 0$ satisfying $B_\rho(x, s) \subseteq B_d(x, r)$. Show that $\mathcal{T}_d$ is weaker than $\mathcal{T}_\rho$. $\lhd$

Before advancing further, and having defined the notion of an open set, we can very well consider the following as the definitions of convergence, closure point, limit point and continuity in the context of a topological space X.

- A sequence (x_n) in X *converges* to $x \in X$ iff for every $A \in \mathcal{G}$ with $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$.
- If $S \subseteq X$, then $x \in X$ is a *closure point* of S iff every $A \in \mathcal{G}$ which contains x contains at least one element of S.
- If $S \subseteq X$, then $x \in X$ is a *limit point* (accumulation point) of S iff every $A \in \mathcal{G}$ which contains x contains at least one element of S other than x.

- If Y also a metric space (possibly different from X), then a function $f : X \to Y$ is *continuous* iff for every open set A in Y, $f^{-1}(A)$ is open in X.

We shall consider the above notions formally in the due course.

Exercise 1.19 Let X be a topological space and $A \subseteq X$. Show that A is open in X iff for every $x \in A$, there exists an open set G such that $x \in G \subseteq A$. $\triangleleft$

Definition 1.20 (Neigbourhood) Given a topological space X and a point $x \in X$, any open set containing $x \in X$ is called an **open neighbourhood** of x. Any set $A \subseteq X$ which contains an open neighbourhood of x is called a **neighbourhood** of x. $\diamondsuit$

Remark 1.21 In some books on topology, neighbourhood of a point $x \in X$ is defined to be any set $A \subseteq X$ which contains an open set containing x. To avoid confusion among readers, we shall use the nomenclature "open neighbourhood" of a point for an open set containing that point. $\diamondsuit$

Remark 1.22 Let X be a set and $\mathcal{S}$ be a family of subsets of X, i.e, $\mathcal{S} \subseteq 2^X$.

- $\mathcal{S}$ is said to be *closed under finite unions* if for any finite number of members of $\mathcal{S}$, say $A_1, \ldots, A_n \in \mathcal{S}$, $\bigcup_{i=1}^n A_i \in \mathcal{S}$.
- $\mathcal{S}$ is said to be *closed under finite intersections* if for any finite number of members of $\mathcal{S}$, say $A_1, \ldots, A_n \in \mathcal{S}$, $\bigcap_{i=1}^n A_i \in \mathcal{S}$.
- $\mathcal{S}$ is said to be *closed under arbitrary unions* if for any sub-collection of $\mathcal{S}$, say $\mathcal{A} \subseteq \mathcal{S}$, we have $\bigcup_{A \in \mathcal{A}} A \in \mathcal{S}$.
- $\mathcal{S}$ is said to be *closed under arbitrary intersections* if for any subcollection of $\mathcal{S}$, say $\mathcal{A} \subseteq \mathcal{S}$, we have $\bigcap_{A \in \mathcal{A}} A \in \mathcal{S}$.

In the above,

- $x \in \bigcup_{A \in \mathcal{A}} A$ iff there exists $A \in \mathcal{A}$ such that $x \in A$;
- $x \in \bigcap_{A \in \mathcal{A}} A$ iff $x \in A$ for all $A \in \mathcal{A}$. $\diamondsuit$

Thus, we see that, if $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is closed under arbitrary unions and finite intersections, and we also have the following result, which can be proved using the definition of the topology.

Results in the following theorem can be verified easily.

Theorem 1.23 *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C}$ be the collection of all closed sets in X. Then*

(i) *$\varnothing$ and X are members of $\mathcal{C}$;*

(ii) *Union of any finite number of members of $\mathcal{C}$ is a member of $\mathcal{C}$.*

(iii) *Intersection of any collection of members of $\mathcal{C}$ is a member of $\mathcal{C}$;*

Exercise 1.24 Write the details of the proof of the above theorem. $\triangleleft$

Notation: Given a set X, if $\mathcal{A}$ is a collection of subsets of X, then we may denote such a collection also by $\{A \subseteq X : A \in \mathcal{A}\}$. Thus, it is of the form $\{A_\alpha \subseteq X : \alpha \in \Lambda\}$, where $\Lambda = \mathcal{A}$, by identifying A_α with A for $\alpha = A$. Thus, an arbitrary collection of subsets of X can always be written as

$$\{A_\alpha \subseteq X : \alpha \in \Lambda\}$$

for some set Λ, which is called an *index set*. Thus, if $\mathcal{A} = \{A_\alpha \subseteq X : \alpha \in \Lambda\}$, then the union of all members of $\mathcal{A}$, written as $\bigcup_{A\in\mathcal{A}} A$ or $\bigcup \mathcal{A}$ is $\bigcup_{\alpha\in\Lambda} A_\alpha$, that is,

$$\bigcup_{A\in\mathcal{A}} A = \bigcup_{\alpha\in\Lambda} A_\alpha.$$

Similarly, the intersection of all members of $\mathcal{A}$, written as $\bigcap_{A\in\mathcal{A}} A$ or $\bigcap \mathcal{A}$ is $\bigcap_{\alpha\in\Lambda} A_\alpha$, that is,

$$\bigcap_{A\in\mathcal{A}} A = \bigcap_{\alpha\in\Lambda} A_\alpha.$$

If $\mathcal{A}$ is a finite set, then we may write it as $\{A_1, \ldots, A_n\}$ or $\{A_i : i \in \{1, \ldots, n\}\}$ for some $n \in \mathbb{N}$, and if $\mathcal{A}$ is a countably infinite set, then it can be written as $\{A_1, A_2, \ldots\}$ or $\{A_n : n \in \mathbb{N}\}$. See also the Appendix.

Note that, if $\mathcal{T}$ is any topology on a set X, and if $\mathcal{T}_0$ and $\mathcal{T}_1$ are the indiscrete topology and discrete topology, respectively, on X, then

$$\mathcal{T}_0 \subseteq \mathcal{T} \subseteq \mathcal{T}_1.$$

Definition 1.25 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. Then, $\mathcal{T}_1$ is said to be **weaker** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and in that case, $\mathcal{T}_2$ is said to be **stronger** than $\mathcal{T}_1$. If $\mathcal{T}_1$ is a proper sub-collection of $\mathcal{T}_2$, then $\mathcal{T}_1$ is said to be **strictly weaker** than $\mathcal{T}_2$, and in that case $\mathcal{T}_2$ is said to be **strictly stronger** than $\mathcal{T}_1$. $\Diamond$

- On a set X, indiscrete topology is the weakest topology and discrete topology is the strongest topology; both coincide if X is a singleton set.

Let us give a few more standard examples which may be of use in the due course.

Example 1.26 Let X be any set.

(i) Let $\mathcal{T}$ consist of $\varnothing$ and all those subsets of X such that A^c is finite. Then $\mathcal{T}$ is a topology on X, called the **co-finite topology** on X.

(ii) Let $\mathcal{T}$ consist of $\varnothing$ and all those subsets A of X such that A^c is countable. Then $\mathcal{T}$ is a topology on X, called the **co-countable topology** on X. $\diamondsuit$

- On a finite set, co-finite topology and co-countable topology are the same. In general, co-finite topology is weaker than co-countable topology; strictly weaker if the underlying set is infinite.
- On $\mathbb{R}^k$, co-finite topology is weaker than the usual topology.
- On $\mathbb{R}^k$, co-countable topology and usual topology are not comparable.

Exercise 1.27 Verify the above statements. $\triangleleft$

Exercise 1.28 Show that the co-finite topology on a nonempty set X is induced by a metric iff X is a finite set, and in that case this topology is the discrete topology. $\triangleleft$

Exercise 1.29 Show that the co-countable topology on a nonempty set X is induced by a metric iff X is a countable set, and in that case this topology is the discrete topology. $\triangleleft$

Example 1.30 Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ such that $A \in \mathcal{T}$ iff for every $x \in A$, there exists $r > 0$ such that $[x, x + r) \subseteq A$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$, called the **lower limit topology** on $\mathbb{R}$. $\diamondsuit$

Notation: In the due course, we shall use the notation $\mathbb{R}_\ell$ for $\mathbb{R}$ with lower limit topology.

Example 1.31 Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ such that $A \in \mathcal{T}$ iff for every $x \in A$, there exists $r > 0$ such that $(x - r, x] \subseteq A$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$, called the **upper limit topology** on $\mathbb{R}$. $\diamondsuit$

Remark 1.32 We observe that every open interval in $\mathbb{R}$ is open with respect to the lower limit topology and upper limit topology. Using this fact, it can be seen that the usual topology on $\mathbb{R}$ is weaker than both lower limit topology and upper limit topology. However, lower limit topology and upper limit topology are not comparable. $\diamondsuit$

Exercise 1.33 Verify the assertions in the above remark. $\triangleleft$

Example 1.34 Let $\mathcal{T} := \{(a,\infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$: To see this, note that $\{\varnothing, \mathbb{R}\} \subseteq \mathcal{T}$ and for every $a, b \in \mathbb{R}$,

$$a < b \quad \Longrightarrow \quad (a,\infty) \cap (b,\infty) = (a,\infty) \in \mathcal{T}.$$

Now consider a collection of sets from $\mathcal{T}$, which is of the form $\{(a,\infty) : a \in S\}$ for some $S \subseteq \mathbb{R}$. Then

$$\bigcup_{a \in S} (a,\infty) = (a_0,\infty) \in \mathcal{T},$$

where $a_0 := \inf S$, which is either in $\mathbb{R}$ or it is $-\infty$. Clearly, $\mathcal{T}$ is weaker than the usual topology on $\mathbb{R}$. $\Diamond$

Theorem 1.35 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on X, which is weaker than $\mathcal{T}_1$ and $\mathcal{T}_2$. More generally, let $\{\mathcal{T}_\alpha : \alpha \in \Lambda\}$ be a family of all topologies on a set X. Then $\bigcap_{\alpha \in \Lambda} \mathcal{T}_\alpha$ is a topology on X which is weaker than $\mathcal{T}_\alpha$ for every $\alpha \in \Lambda$.*

Proof. Let $\mathcal{T} := \bigcap_{\alpha \in \Lambda} \mathcal{T}_\alpha$. Since each $\mathcal{T}_\alpha$ contains $\varnothing$ and X, $\mathcal{T}$ contains $\varnothing$ and X. Now, let $A, B \in \mathcal{T}$. Then $A \in \mathcal{T}_\alpha$ and $B \in \mathcal{T}_\alpha$ for every $\alpha \in \Lambda$. Hence, $A \cap B \in \mathcal{T}_\alpha$ for every $\alpha \in \Lambda$, and hence $A \cap B \in \mathcal{T}$. Next, let $\mathcal{A}$ be a sub-collection of $\mathcal{T}$. Then $\mathcal{A}$ is a sub-collection of $\mathcal{T}_\alpha$ for every $\alpha \in \Lambda$. Hence, union of members of $\mathcal{A}$ belongs to $\mathcal{T}_\alpha$ for every $\alpha \in \Lambda$. Therefore, union of members of $\mathcal{A}$ belongs to $\mathcal{T}$. Thus, $\mathcal{T}$ is a topology on X. Clearly, $\mathcal{T}$ is weaker than $\mathcal{T}_\alpha$ for each $\alpha \in \Lambda$. $\square$

We know that an arbitrary collection $\mathcal{S}$ of subsets of a set X need not be a topology on X. However, there is a smallest topology on X containing $\mathcal{S}$, a the following theorem shows.

Theorem 1.36 *Let $\mathcal{S}$ be a collection of subsets of X. Then the intersection of all topologies on X containing $\mathcal{S}$ is a topology on X, and it is the weakest topology containing $\mathcal{S}$.*

Proof. By Theorem 1.35, if $\mathcal{T}$ is the intersection of all topologies containing $\mathcal{S}$, then $\mathcal{T}$ is a topology on X such that $\mathcal{S} \subseteq \mathcal{T}$. If $\widetilde{\mathcal{T}}$ is any topology containing $\mathcal{S}$, then $\mathcal{T} \subseteq \widetilde{\mathcal{T}}$, so that $\mathcal{T}$ is the weakest topology on X containing the collection $\mathcal{S}$. $\square$

Definition 1.37 Let $\mathcal{S}$ be a collection of subsets of X. The weakest topology on X containing $\mathcal{S}$ is called the **topology generated by** $\mathcal{S}$. $\Diamond$

Example 1.38 The usual topology on $\mathbb{R}$ is generated by the following collection of sets:

(i) The collection of all open intervals of the form (a, b) with $a < b$.

(ii) The collection of all open intervals of the form (a, ∞) and $(-\infty, b)$.

The result in (i) follows, because every open set is a union of intervals of the form (a, b), and (ii) follows from (i), since for $a < b$, we have $(a, b) = (-\infty, b) \cap (a, \infty)$, $\Diamond$

Example 1.39 Let $\mathcal{S}$ be the collection of all intervals of the form $[a, b)$ with $a < b$. Then the topology generated by $\mathcal{S}$ is the lower limit topology on $\mathbb{R}$. $\Diamond$

Chapter 2

Basic Notions and Properties

In this chapter, we extend the foundational concept of convergence from metric spaces to the more general setting of topological spaces. Additionally, we introduce the notion of a base for a topology. Acting as the fundamental building blocks of a topological space, a base is the direct analogue to the family of all open balls in a metric space, allowing us to generate complex topologies from simpler, foundational collections of open sets.

2.1 Convergence

Definition 2.1 (Convergence in topological space) Let $(X, \mathcal{T})$ be a topological space. A sequence (x_n) in X is said to **converge** to $x \in X$ if for every open set G containing x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in G$ for all $n \geq n_0$. If (x_n) converges to x, then we write

$$x_n \to x \quad \text{or} \quad \lim_{n \to \infty} x_n = x,$$

and in that case x is called a **limit** of (x_n). ◊

> If X is a metric space with metric d, then for a given sequence (x_n) in X, $x_n \to x$ with respect to $\mathcal{T}_d$ iff $d(x_n, x) \to 0$

- In a metric space, every convergent sequence has a unique limit. However, this need not be true in every topological space, as the following example shows.

Example 2.2 In an indiscrete topological space X, every sequence converges to every point in X, as X is the only nonempty open set. In particular, if X contains more than one point, then every sequence in X has more than one limit. ◊

Exercise 2.3 In a discrete topological space every convergent sequence is eventually constant. ◁

Example 2.4 Let X be a family of real valued functions defined on an interval $I := [a,b]$. (For example $X = C([a,b],\mathbb{R})$.) Let $t_0 \in [a,b]$ and for $f \in X$ and $r > 0$, let

$$E_r(f) := \{g \in X : |g(t_0) - f(t_0)| < r\}.$$

Let $\mathcal{T}$ be a class of subsets of X such that $A \in \mathcal{T}$ iff for every $f \in A$, there exists $r > 0$ such $E_r(f) \subseteq A$. Then, $\mathcal{T}$ is a topology on X. We note that, if (f_n) is a sequence in X and $f \in X$, then with respect to this topology,

$$f_n \to f \iff f_n(t_0) \to f(t_0).$$

In particular, if (f_n) is in X such that $(f_n(t_0))$ converges, and if $\alpha := f_n(t_0)$, then (f_n) converges to any function $g \in X$ for which $g(t_0) = \alpha$. Thus, the limit function f is not unique. ◊

One of the properties that a topological space may have which ensures unique limit for convergent sequences is *Hausdorff property*.

Definition 2.5 (Hausdorff space) A topological space X is said to be a **Hausdorff**[1] **space** if for every pair of distinct points x, y in X, there exist disjoint open sets A and B containing x and y, respectively. ◊

- Every metrizable space is Hausdorff.

Theorem 2.6 *In a Hausdorff space, every convergent sequence has a unique limit.*

Proof. Let X be a Hausdorff space and (x_n) be a convergent sequence in X. Suppose $x, y \in X$ such that $x_n \to x$ and $x_n \to y$. We have to show that $x = y$.

Assume, on the contrary, that $x \neq y$. Then by the Hausdorff property, there exist disjoint open sets A and B such that $x \in A$ and $y \in B$. Since $x_n \to x$ and $x_n \to y$, there exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that

$$x_n \in A \quad \forall n \geq n_1 \quad \text{and} \quad x_n \in B \quad \forall n \geq n_2.$$

Then we have

$$x_n \in A \cap B \quad \forall n \geq n_0 := \max\{n_1, n_2\}.$$

This is not possible, since A and B are disjoint. Hence, our assumption that $x \neq y$ is wrong. Thus, we have proved that $x = y$. □

[1]Felix Hausdorff (1868-1942) was a mathematician from Germany.

- The topological space in Example 2.4 is not Hausdorff.

Example 2.7 Let X be a set. The following can be verified easily (Verify!):

1. The indiscrete topology on X is Hausdorff iff $\#(X) \leq 1$, and in that case the topology is the discrete topology.
2. The co-finite topology on X is Hausdorff iff X is a finite set, and in that case the topology is the discrete topology.
3. The co-countable topology on X is Hausdorff iff X is a countable set, and in that case the topology is the discrete topology. ♢

Example 2.8 The lower limit topology and the upper limit topology on $\mathbb{R}$ are Hausdorff: First consider $\mathbb{R}_\ell$, the set $\mathbb{R}$ with lower limit topology. Let $x, y \in \mathbb{R}_\ell$ with $x \neq y$. Taking $r = |x - y|/2$, we see that the open sets $A := [x, x + r)$ and $B := [y, y + r)$ disjoint open sets in $\mathbb{R}_\ell$ containing x and y, respectively. Thus, we have shown that $\mathbb{R}_\ell$ is a Hausdorff space. Similarly, it can be shown that $\mathbb{R}$ with upper limit topology is also Hausdorff. ♢

Remark 2.9 We have seen in Example 2.8 that the lower limit topology and the upper limit topology on $\mathbb{R}$ are Hausdorff. However, they are not metrizable. We shall prove this fact in the next chapter (see Theorem 3.9). ♢

Example 2.10 For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define

$$\rho(x, y) = |x_1 - y_1|, \quad x, y \in \mathbb{R}^2.$$

Note that, $\rho(\cdot, \cdot)$ is a pseudo metric[2]. For $x \in X$ and $r > 0$, let

$$B_\rho(x) := \{u \in \mathbb{R}^2 : \rho(x, u) < \rho\}.$$

Then

$$\mathcal{T} := \{A \subseteq \mathbb{R}^2 : \forall\, x \in \mathbb{R}^2, \exists\, r > 0 \text{ such that } B_\rho(x) \subseteq A\}$$

is a topology on $\mathbb{R}^2$, which is not Hausdorff. ♢

Exercise 2.11 Verify the assertions in Example 2.10. ◁

Exercise 2.12 Show that the topology on $\mathbb{R}^2$ defined as in Example 2.10 is weaker than the usual topology on $\mathbb{R}^2$. ◁

[2]A function $\rho : X \times X \to \mathbb{R}$ is said to be a semi-metric or *pseudo-metric* if it satisfies all the properties of a metric, except that $\rho(x, y) = 0$ need not imply $x = y$.

- We have seen that Hausdorff property is a sufficient condition for unique limit for every convergent sequence. However, it is not a necessary condition, as the following theorem shows.

Theorem 2.13 *Every convergent sequence in a co-countable topological space is eventually constant.*

Proof. Let X be a co-countable topological space. If X is a countable set, then co-countable topology on X is the discrete topology, and hence every convergent sequence in X is eventually constant. So, assume that X is an uncountable set. Let (x_n) in X be such that $x_n \to x$. Let $A := \{x_n : x_n \neq x\}$. Since A is countable, A^c is an uncountable open set containing x. Hence, there exists $k \in \mathbb{N}$ such that $x_n \in A^c$ for all $n \geq k$. Thus, $x_n = x$ for all $n \geq k$. □

- Every convergent sequence in a co-countable topological space has a unique limit.

A convergent sequence in a co-finite topological space need not be eventually constant, as the following example shows.

Example 2.14 Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with co-finite topology. Consider the sequence $(1/n)$. Let G be an open subset of X containing 0. Then $X \setminus G$ is a finite set. Hence, there exists $k \in \mathbb{N}$ such that $\frac{1}{n} \in G$ for all $n \geq k$. Hence, $\frac{1}{n} \to 0$. This shows that, a convergent sequence in a co-finite topological space need not be eventually constant. ♢

- Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X such that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$. Then every sequence in X, which converges with respect to $\mathcal{T}_2$ is convergent with respect to $\mathcal{T}_1$, and the converse need not be true.

The following example illustrates the second part of the assertion in the above example using the usual topology and lower topology on $\mathbb{R}$.

Example 2.15 Let $X = \mathbb{R}_\ell$, that is $\mathbb{R}$ with lower limit topology. Recall from Remark 1.32 that the usual topology on $\mathbb{R}$ is weaker than the lower limit topology. We observe that, for a sequence (x_n) in X, $x_n \to x$ iff for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$x_n \in [x, x+\varepsilon) \quad \forall\, n \geq N.$$

Consider the sequences $(1/n)$ and $(-1/n)$ in $\mathbb{R}$. Each of these sequences converge to 0 with respect to the usual topology, and $(1/n)$ converges to 0 in $\mathbb{R}_\ell$. However,

$$-\frac{1}{n} \not\to 0.$$

Further, $(x_n) := (-1/n)$ does not converge to any point in $\mathbb{R}_\ell$: To see this, let $x \in \mathbb{R}_\ell$. If $x \geq 0$, then the open set $[x, x+1)$ in $\mathbb{R}_\ell$ does not contain any members of the sequence, so that $x_n \not\to x$ in $\mathbb{R}_\ell$. Also, if $x < 0$, taking $0 < \varepsilon < |x|$, the open set $[x, x+\varepsilon)$ can contain only a finite number of members of the sequence, so that $x_n \not\to x$ in $\mathbb{R}_\ell$. $\Diamond$

Example 2.16 Let $\mathcal{T}$ be the topology on $\mathbb{R}$ defined as in Example 1.34. We have seen that this topology is weaker than the usual topology. Thus, every sequence which converges with respect to the usual topology converges with respect to $\mathcal{T}$ as well. We note that:

> If (x_n) is any sequence of real numbers which is bounded below and if $x \geq x_0$, where $x_0 = \inf_n x_n$, then $x_n \to x$ with respect to $\mathcal{T}$.

Indeed, if G is any open set containing x, then G is of the form $G = (a, \infty)$ for some $a < x$ so that $x_n \in G$ for all $n \in \mathbb{N}$.

> If (x_n) is any increasing sequence of real numbers and $x \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $x \in [x_1, x_k]$, then $x_n \to x$ with respect to $\mathcal{T}$.

Indeed, if G is any open set containing x, then G is of the form $G = (a, \infty)$ for some $a \in \mathbb{R}$, and hence $x_n \in G$ for all $n \geq k$. $\Diamond$

2.2 Limit points, closure, interior and boundary

Definition 2.17 Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

1. A point $x \in X$ is said to be a **limit point** of A if every open set containing x contains at least one point of A other than x. The set of all limit points of A is called the **derived set** of A, and it is denoted by A'.

2. A point $x \in X$ is said to be an **interior point** of A if there exists an open set containing x and contained in A. The set of all interior points of A is called the **interior** of A, and it is denoted by A° or $\mathrm{int}(A)$.

3. A point $x \in X$ is said to be a **closure point** of A if every open set containing x contains some point of A. The set of all closure points of A called the **closure** of A, and it is denoted by $\bar{A}$ or $\mathrm{cl}\,(A)$.

4. A point $x \in X$ is said to be a **boundary point** of A if open set containing x contains some point of A and some point of A^c. The set of all boundary points of A is called the **boundary** of A, and it is denoted by ∂A or $\mathrm{bd}(A)$.

5. A subset D of X is said to be dense in X if $\mathrm{cl}\,(D) = X$. $\Diamond$

Theorem 2.18 *Let A be a subset of a topological space X. Then*

$$A^\circ \subseteq A \subseteq \bar{A} = A \cup A' = A \cup \partial A = A^\circ \cup \partial A.$$

Proof. It can be easily seen that

$$A^\circ \subseteq A \subseteq \bar{A} = A \cup A' = A \cup \partial A.$$

To see $\bar{A} = A^\circ \cup \partial A$, first we observe that $A^\circ \cup \partial A \subseteq \bar{A}$. To see the other way inclusion, let $x \in \bar{A}$. Then, every open set G containing x contains some some point of A. If G does not contain any point of A^c, then $G \subseteq A$ so that $x \in A^\circ$. Thus, we have proved that if $x \notin \partial A$, then $x \in A^\circ$. in other words, we have proved that $\bar{A} \subseteq A^\circ \cup \partial A$. $\square$

If A is a subset of a topological space X, then

$A = \bar{A}$ iff A contains all its limit points
iff A contains all its boundary points.

The reader is urged to verify the assertions in the following three examples.

Example 2.19 Let $\mathbb{R}$ be with usual topology. Then we have the following:

1. If $A = (0, 1]$, then $A^\circ = (0, 1)$, $\quad \bar{A} = [0, 1]$, $\quad A' = [0, 1]$, $\quad \partial A = \{0, 1\}$.
2. If $A \in \{\mathbb{Q}, \mathbb{Q}^c\}$, then $A^\circ = \varnothing$, $\quad \bar{A} = \mathbb{R}$, $\quad A' = \mathbb{R}$, $\quad \partial A = \mathbb{R}$.
3. If $A \in \{\mathbb{N}, \mathbb{Z}\}$, then $A^\circ = \varnothing$, $\quad \bar{A} = A$, $\quad A' = \varnothing$, $\quad \partial A = \varnothing$. $\Diamond$

Example 2.20 Let $\mathbb{R}^2$ be with usual topology.

(i) Let S be the *unit circle*, that is,

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}.$$

Then we have

$$S^\circ = \varnothing, \quad \bar{S} = S, \quad S' = S, \quad \partial S = S.$$

(ii) Let A be the *open unit unit disc*, and B be the *closed unit disc*, that is,

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}, \quad B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

Then (Verify), we have

$$A^\circ = A, \quad \bar{A} = B, \quad A' = B, \quad \partial A = S,$$

$$B^\circ = A, \quad \bar{B} = B, \quad B' = B, \quad \partial B = S.$$

$\Diamond$

Example 2.21 Let $\mathbb{R}$ be with co-countable topology. Note that, with respect to this topology, any open subset has to be an uncountable set, and every countable set is closed. Hence,

$$\bar{\mathbb{Q}} = \mathbb{Q}, \quad \mathbb{Q}^\circ = \varnothing, \quad \overline{\mathbb{Q}^c} = \mathbb{R}, \quad (\mathbb{Q}^c)^\circ = \mathbb{Q}^\circ. \qquad \Diamond$$

Example 2.22 Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, X, \{a\}\}$. Then

1. If $A = \{a\}$, then $A^\circ = A, \quad \bar{A} = X, \quad A' = \{b, c\}$.
2. If $A = \{b\}$, then $A^\circ = \varnothing, \quad \bar{A} = \{b, c\}, \quad A' = \{c\}$.
3. If $A = \{c\}$, then $A^\circ = \varnothing, \quad \bar{A} = \{b, c\}, \quad A' = \{b\}$.
4. If $A = \{a, b\}$, then $A^\circ = \{a\}, \quad \bar{A} = X, \quad A' = \{b, c\}$.
5. If $A = \{a, c\}$, then $A^\circ = \{a\}, \quad \bar{A} = X, \quad A' = \{b, c\}$.
6. If $A = \{b, c\}$, then $A^\circ = \varnothing, \quad \bar{A} = A = A'$. $\Diamond$

Exercise 2.23 Verify the assertions in the above three examples. $\triangleleft$

Theorem 2.24 *Let X be a topological space, and A and B be subsets of X such that $A \subseteq B$. Then*

$$A^\circ \subseteq B^\circ, \quad \bar{A} \subseteq \bar{B}.$$

Proof. Suppose $x \in A^\circ$. Then there exists an open set G such that $x \in G \subseteq A$. Since $A \subseteq B$, this shows that $x \in B^\circ$. Thus, $A^\circ \subseteq B^\circ$.

Next, suppose $x \in \bar{A}$. Then, for every open set G containing x, $G \cap A \neq \varnothing$, so that we also have $G \cap B \neq \varnothing$, showing that $x \in \bar{B}$. Thus, $\bar{A} \subseteq \bar{B}$. $\square$

Theorem 2.25 *Let X be a topological space and $A \subseteq X$.*

(1) *$\bar{A}$ is a closed set.*

(2) *A° is an open set.*

(3) *A is a closed set iff $\bar{A} = A$.*

(4) *A is open iff $A^\circ = A$.*

Proof. (1). To show that $(\bar{A})^c$ is an open set. If $\bar{A} = X$, then its compliment is $\varnothing$, which is an open set. So, assume that $\bar{X} \neq X$. Let $x \in (\bar{A})^c$, that is, x is not a closure point of A. Then there exists an open set G_x containing x, which does not contain any point of A. Also, if $y \in G_x$, then $y \notin \bar{A}$, since the open set G_x containing y does not contain any point of A. Thus, $G_x \cap \bar{A} = \varnothing$, so that $G_x \subseteq (\bar{A})^c$. Therefore,

$$(\bar{A})^c \subseteq \bigcup_{x \in (\bar{A})^c} G_x \subseteq (\bar{A})^c.$$

Hence, $(\bar{A})^c$ is an open set.

(2). Let $x \in A^\circ$. Then there exists an open set G_x containing x such that $G_x \subseteq A$. This also implies that $G_x \subseteq A^\circ$, since for every $y \in G_x$, G_x is an open set containing y and contained in A. Hence,

$$A^\circ \subseteq \bigcup_{x \in A^\circ} G_x \subseteq A^\circ.$$

Thus, A° is an open set.

(3). By (1), if $\bar{A} = A$, then A is a closed set. Conversely, suppose that A is a closed set. We have to prove that $\bar{A} = A$. Since $A \subseteq \bar{A}$, it is enough to prove that $\bar{A} \subseteq A$. That is, to prove that $x \notin A \Longrightarrow x \notin \bar{A}$. Suppose $x \notin A$, that is, $x \in A^c$. Thus, A^c is an open set containing x which does not contain any point of A. Hence, $x \notin \bar{A}$.

(4). By (2), if $A^\circ = A$, then A is an open set. Conversely, suppose that A is an open set. We have to prove that $A^\circ = A$. Since $A^\circ \subseteq A$, it is enough to prove $A \subseteq A^\circ$. Let $x \in A$. Since A is an pen set containing x and contained in A, x is an interior point of A. Thus, $A \subseteq A^\circ$. □

Theorem 2.26 *Let X be a topological space $A \subseteq X$. The following results hold.*

(1) *A is open iff A does not contain any of its boundary points.*

(2) *A° is the largest open set contained in A.*

(3) *$\bar{A}$ is the smallest closed set containing A.*

Proof. (1) It is enough to show that A is not open iff A contains a boundary point of A. First we recall that A is open iff $A = A^\circ$.

Suppose that A is not open. Then there exists $x \in A \setminus A^\circ$, which implies that every open neighbourhood of x contains some point from A^c. This shows that x is a boundary point of A.

Conversely, suppose A contains a boundary point x of A. Then every open neighbourhood of x contains a point from A^c so that $x \notin A^\circ = A$, so that A is not open.

(2) We know that A° is an open set. Suppose G is an open set such that

$G \subseteq A$. Then we have $G = G^\circ \subseteq A^\circ$. Thus, A° is the largest open set contained in A.

(3) We know that $\bar{A}$ is a closed set. Suppose F is a closed set such that $A \subseteq F$. Then we have $\bar{A} \subseteq \bar{F} = F$. Thus, $\bar{A}$ is the smallest closed set containing A. □

Exercise 2.27 If X is with discrete topology, then for any $A \subseteq X$, $A^\circ = A = \bar{A}$. ◁

Definition 2.28 (Isolated point) Let X be a topological space and $A \subseteq X$. A point $x \in A$ is called an **isolated point** if it is not a limit point of A. ♢

Theorem 2.29 *Let X be a topological space and $A \subseteq X$. Then a point $x \in A$ is an isolated point of A iff the singleton set $\{x\}$ is open in A.*

Proof. We observe that $x \in A$ is an isolated point of A iff there exists an open set V in X which does not conain any other point of A iff there exists an open set V in X such that $V \cap A = \{x\}$ iff $\{x\}$ is open is open in A □

Corollary 2.30 *Let X be a topological space. Then every point in X is an isolated point iff X is a discrete space.*

- Let X be a topological space and $A \subseteq X$. If (x_n) is sequence in A consisting of distinct points which converges to $x \in X$, then x is limit point of A. However, a limit point of A need not be a limit of any sequence in A, as the following example shows.

Example 2.31 Let $\mathbb{R}$ be with co-countable topology. Then every rational number is a limit point of $\mathbb{R}\setminus\mathbb{Q}$. However, in view of Theorem 2.13, a sequence in $\mathbb{R} \setminus \mathbb{Q}$ cannot converge to any rational number. ♢

Theorem 2.32 *Suppose X is a Hausdorff space and $A \subseteq X$. Then a point $x \in X$ is a limit point of A iff every open neighbourhood of x contains infinite number of elements from A.*

Proof. Suppose $x \in A'$. Assume for a moment that there exist an open neighbourhood G of x such that $G \cap A$ contains only finite number of members. Since $x \in A'$, we know that $G \cap (A \setminus \{x\}) \neq \varnothing$. Suppose

$$G \cap (A \setminus \{x\}) = \{x_1, \ldots, x_n\}$$

for some $n \in \mathbb{N}$. Since X is Hausdorff, for each $i \in \{1, \ldots, n\}$, there exist disjoint open sets U_i abd V_i such that $x \in U_i$ and $x_i \in V_i$. Let $U := \bigcap_{i=1}^n U_i$. Then we have

$$x \in U \cap G, \quad \{x_1, \ldots, x_n\} \cap U = \varnothing,$$

so that $U \cap (G \cap (A \setminus \{x\}) = \varnothing$. This is a contradiction to the fact that the open set $U \cap G$ must contain some point from $A \setminus \{x\}$.

The converse is obvious from the definition of a limit point. $\square$

Definition 2.33 (Nowhere dense) Let X be a topological space and $A \subseteq X$. Then A is said to be **nowhere dense** in X if $(\bar{A})^\circ = \varnothing$. $\Diamond$

Example 2.34 Let $X = \mathbb{R}$ with usual topology. Then $\mathbb{Z}$ is nowhere dense in X since $\mathbb{Z}$ is closed in X and $\mathbb{Z}^\circ = \varnothing$, and $\mathbb{Q}$ is not nowhere dense since $(\bar{A})^\circ = \mathbb{R}^\circ \neq \varnothing$. $\Diamond$

Theorem 2.35 *Let X be a topological space and A be a closed set in X. Then A is nowhere dense in X iff A^c is dense in X.*

Proof. We observe that A is nowhere dense in X iff $A^\circ = \varnothing$ iff A does not contain any nonempty open set iff every nonempty open set contains some point from A^c iff A^c is dense in X. $\square$

Remark 2.36 In Theorem 2.35, the condition that A is closed cannot be dropped. To see this, consider $X = \mathbb{R}$ with usual topology and $A = \mathbb{Q}$. Then we know that $A^c = \mathbb{R} \setminus \mathbb{Q}$ dense, but $(\bar{A})^\circ = \mathbb{R}^\circ = \mathbb{R}$. $\Diamond$

2.3 Subspace topology

Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. Then the topology $\mathcal{T}$ induces a topology on Y in a natural way as per the following theorem.

Theorem 2.37 *Let $(X, \mathcal{T})$ be a topological space, and $Y \subseteq X$. Then*

$$\mathcal{T}_Y := \{G \cap Y : G \in \mathcal{T}\}$$

is a topology on Y.

Proof. Clearly $\varnothing \in \mathcal{T}_Y$. Since $\in \mathcal{T}$, $Y = X \cap Y \in \mathcal{T}_Y$. Let $A_1, A_2 \in \mathcal{T}_Y$. Then there exist $G_1, G_2 \in \mathcal{T}$ such that

$$A_1 = G_1 \cap Y, \quad A_2 = G_2 \cap Y$$

so that

$$A_1 \cap A_2 = (G_1 \cap G_2) \cap Y \in \mathcal{T}_Y.$$

Next, suppose $\{A_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{T}_Y$. Then, for each $\alpha \in \Lambda$, there exists $G_\alpha \in \mathcal{T}$ such that $A_\alpha = G_\alpha \cap Y$. Hence,

$$\bigcup_{\alpha\in\Lambda} A_\alpha = \Big(\bigcup_{\alpha\in\Lambda} G_\alpha\Big) \cap Y \in \mathcal{T}_Y.$$

Thus, we have proved that $\mathcal{T}_Y$ is a topology on Y. $\square$

Definition 2.38 (Subspace topology) The topology $\mathcal{T}_Y$ obtained in Theorem 2.37 is called the **subspace topology** on Y or the **topology on** Y **induced by** $\mathcal{T}$. $\Diamond$

Convention: When we say that a subset A of Y is open in Y, it means that it is open with respect to the subspace topology on Y.

- Let X be a topological space and $A \subseteq Y \subseteq X$. If A is open in Y, then A need not be open in X.

Example 2.39 Let $\mathbb{R}$ be with usual topology .

(i) let $Y = [0, 1]$. Then $[0, 1/2)$ is open in Y, but $[0, 1/2)$ is not open in $\mathbb{R}$.

(ii) Let $Y = \mathbb{Z}$. Then $\mathcal{T}_Y$ is the discrete topology on $\mathbb{Z}$, since for every $n \in \mathbb{Z}$, we have $\{n\} = \mathbb{Z} \cap (n - \frac{1}{2}, n + \frac{1}{2})$, so that every subset of Y is open in Y. Note that, $\varnothing$ is the only subset of Y which is open in X. $\Diamond$

However, if $Y \subseteq X$ is open in X, then every open set in Y is open in X. In fact, we have the following theorem.

Theorem 2.40 *Let X be a topological space and $Y \subseteq X$. Then,*

$$\mathcal{T}_Y \subseteq \mathcal{T} \iff Y \in \mathcal{T},$$

and in that case $\mathcal{T}_Y = \{A \subseteq Y : A \in \mathcal{T}\}$. In other words, every open set in Y is open in X iff Y is open in X.

Proof. Suppose $\mathcal{T}_Y \subseteq \mathcal{T}$. Since $Y \in \mathcal{T}_Y$, we obtain $Y \in \mathcal{T}$. Conversely, suppose $Y \in \mathcal{T}$. If $A \in \mathcal{T}_Y$, then $A = G \cap Y$ for some $G \in \mathcal{T}$; and since $Y \in \mathcal{T}$, we obtain $A = G \cap Y \in \mathcal{T}$. Thus, $\mathcal{T}_Y \subseteq \mathcal{T}$. $\square$

- Let X be a topological space and $Y \subseteq X$. By Theorem 2.40, every open set in Y is open in X iff Y is open in X.

- An open set in Y can be open in X, without Y being open in X, as the following example shows.

Example 2.41 Let $X = \mathbb{R}$ be with usual topology and $Y = [0,1]$ be with subspace topology. Then the set $A := (0,1)$ is open in Y and X, though Y is not open in X. $\Diamond$

In fact, we have the following theorem.

Theorem 2.42 *Let $(X, \mathcal{T})$ be a topological space, and $Y \subseteq X$. Then*

$$\{A \subseteq Y : A \in \mathcal{T}\} \subseteq \mathcal{T}_Y.$$

Further, equality holds above iff Y is open in X, that is,

$$\{A \subseteq Y : A \in \mathcal{T}\} = \mathcal{T}_Y \iff Y \in \mathcal{T}.$$

In particular, if Y is open in X, then a subset of Y is open in Y iff it is open in X.

Proof. If $A \subseteq Y$ and $A \in \mathcal{T}$, then, since $A = A \cap G$ with $G = A \in \mathcal{T}$, so that $A \in \mathcal{T}_Y$. Thus, we obtain $\{A \subseteq Y : A \in \mathcal{T}\} \subseteq \mathcal{T}_Y$.

Next, suppose that $\{A \subseteq Y : A \in \mathcal{T}\} = \mathcal{T}_Y$. Then, $Y \in \mathcal{T}$ since $Y \in \mathcal{T}_Y$. Conversely, suppose that $Y \in \mathcal{T}$ and $A \in \mathcal{T}_Y$. Then since $A = Y \cap G$ for some $G \in \mathcal{T}$, we obtain $A \in \mathcal{T}$. Hence, we have $\mathcal{T}_Y \subseteq \{A \subseteq Y : A \in \mathcal{T}\}$. Since we already have $\{A \subseteq Y : A \in \mathcal{T}\} \subseteq \mathcal{T}_Y$, we obtain $\{A \subseteq Y : A \in \mathcal{T}\} = \mathcal{T}_Y$. $\square$

Exercise 2.43 Let X be a topological space, Y is open in X and $A \subseteq Y \subseteq X$. Show that A is open in Y iff A is open X. $\triangleleft$

Remark 2.44 Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. It can be easily seen that

$$\widetilde{\mathcal{T}}_Y := \{Y\} \cup \{A \subseteq Y : A \in \mathcal{T}\}\}$$

is a topology on Y. By Theorem 2.42, we know that this topology is weaker than $\mathcal{T}_Y$. It can be too weak! For example, let $X = \mathbb{R}$ with usual topology and $Y = \mathbb{Q}$. In this case we know that the only subset of $\mathbb{Q}$ which is open in $\mathbb{R}$ is $\varnothing$. Thus, $\widetilde{\mathcal{T}}_Y = \{\varnothing, \mathbb{Q}\}$, the indiscrete topology on $\mathbb{Q}$, whereas $\mathcal{T}_Y$ contains all sets of the form $G \cap \mathbb{Q}$ where G is open in $\mathbb{R}$. $\Diamond$

Theorem 2.45 *Let X be a topological space, Y be a closed subset of X and $A \subseteq Y$. Then A is closed in Y iff A is closed in X.*

Proof. Suppose Y is closed in X and A is closed in Y. We observe that

$$X \setminus A = (X \setminus Y) \cup (Y \setminus A).$$

Since $Y\setminus A$ is open in Y, there exists an open set G in X such that $Y\setminus A = Y\cap G$. Hence,

$$X \setminus A = (X \setminus Y) \cup (Y \cap G) = Y^c \cup (Y \cap G) = (Y^c \cup Y) \cap (Y^c \cap G) = Y^c \cap G.$$

Thus, $X \setminus A$ is open in X.

Conversely, suppose Y is closed in X and $A \subseteq Y$ is closed in X. Since $Y \setminus A = Y \cap A^c$, where A^c is open in X, we obtain that $Y \setminus A$ is open in Y, so that A is closed in Y. □

Theorem 2.46 *Let X be a topological space and $Y \subseteq X$. Let $\mathcal{C}_X$ and $\mathcal{C}_Y$ denote the set of all closed sets in X and Y, respectively. Then,*

$$\mathcal{C}_Y \subseteq \mathcal{C}_X \iff Y \in \mathcal{C}_X.$$

In other words, every closed set in Y is closed in X iff Y is closed in X.

Proof. Suppose $\mathcal{C}_Y \subseteq \mathcal{C}_X$. Since $Y \in \mathcal{C}_Y$, we obtain $Y \in \mathcal{C}_X$. Conversely, suppose $Y \in \mathcal{C}_Y$. Then, by Theorem 2.45, we obtain $\mathcal{C}_Y \subseteq \mathcal{C}_X$. □

Finally we also note the following.

Theorem 2.47 *Let X be a topological space and $Y \subseteq X$. Let $A \subseteq Y$. Then A is closed in Y iff there exists a closed set F in X such that $A = F \cap Y$.*

Proof. Suppose A is closed in Y. Then, $Y \setminus A$ is open in Y so that $Y \setminus A = G \cap Y$ for some open set G in X. Note that

$$Y \setminus A = G \cap Y \iff (Y \cap A^c)^c = G^c \cup Y^c \iff Y^c \cup A = G^c \cup Y^c.$$

Intersecting with Y, we obtain $A = Y \cap G^c$, where G^c is closed in X. Thus, there exists a closed set F in X such that $A = F \cap Y$.

Conversely, suppose that there exists a closed set F in X such that $A = F \cap Y$. To show that A is closed in Y. Note that

$$Y \setminus A = Y \setminus F \cap Y = Y \cap (F \cap Y)^c = Y \cap (F^c \cup Y) = Y \cap F^c.$$

Thus, $Y \setminus A$ is open in Y. □

2.4 Base and sub-base

Bases are, in some sense, the building blocks for topological spaces. Recall that, in a metric space X, a subset A of X is open iff for every $x \in A$, there exists an open ball centered at x and contained in A. This is equivalent to saying that, every open set is a union of open balls. Motivated by this, we define the notion of a base for a topology.

Definition 2.48 (Base) Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a sub-collection of $\mathcal{T}$. Then $\mathcal{B}$ is called a **base** or **basis** for $\mathcal{T}$ if every member of $\mathcal{T}$ is a union of some (countable or uncountable) members of $\mathcal{B}$. $\diamondsuit$

Here is a very useful characterization of a base.

Theorem 2.49 *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a sub-collection of $\mathcal{T}$. Then $\mathcal{B}$ is a base for $\mathcal{T}$ iff for every $G \in \mathcal{T}$ and for every $x \in G$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq G$.*

Proof. Suppose $\mathcal{B}$ is a base for $\mathcal{T}$ and $G \in \mathcal{T}$. Then there exists $\{B_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{B}$ for some index set Λ such that $G = \bigcup_{\alpha \in \Lambda} B_\alpha$. Hence, for every $x \in G$, there exists $\alpha \in \Lambda$ such that $x \in B_\alpha \subseteq G$.

Conversely, $G \in \mathcal{T}$ and that for every $x \in G$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq G$. Then we ee that $G = \bigcup_{x \in G} B_x$. $\square$

- For every topological space, the the topology itself is a base for the topology.

Example 2.50 Consider the set $\mathbb{R}$.

(1) Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) with $a < b$. Then $\mathcal{B}$ is a base for the usual topology on $\mathbb{R}$.

(2) Let $\mathcal{B}$ be the collection of all intervals of the form $[a, b)$ with $a < b$. Then $\mathcal{B}$ is a base for the lower limit topology on $\mathbb{R}$.

(3) Let $\mathcal{B}$ be the collection of all intervals of the form $(a, b]$ with $a < b$. Then $\mathcal{B}$ is a base for the lower limit topology on $\mathbb{R}$. $\diamondsuit$

Example 2.51 If X is a metric space with metric d, then the collection of all open balls in X is a base for $\mathcal{T}_d$.

If X is a discrete topological space then the singleton sets are open balls and the set of all singleton sets is a base for the topology. $\diamondsuit$

Example 2.52 If X is an indiscrete topological space then $\{X\}$ and $\{\varnothing, X\}$ are the only bases for the topology. $\diamondsuit$

Theorem 2.53 *Suppose $\mathcal{B}$ is a base for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X. Then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proof. Let $A \in \mathcal{T}_1$. Then A is a union of members of $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{T}_2$ as well, it follows that $A \in \mathcal{T}_2$. Thus, $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Similarly, $\mathcal{T}_2 \subseteq \mathcal{T}_1$. $\square$

Suppose X is a set. Then a given collection of subsets of X need not form a base for some topology on X. However, we have the following two theorems.

Theorem 2.54 *Let X be a set and $\mathcal{B}$ be a family of subsets of X such that $\mathcal{B}$ covers X and $\mathcal{B}$ is closed under finite intersections. Then $\mathcal{B}$ is a base for a (unique) topology on X.*

Proof. Let $\mathcal{T}$ be the family of all unions of members of $\mathcal{B}$ together with the empty set. Then, $\mathcal{T}$ is a topology for which $\mathcal{B}$ is a base. Further, by Theorem 2.53, this is the only topology for which $\mathcal{B}$ is a base. $\square$

Recall that, if (X, d) is a metric space and $\mathcal{B} := \{B(x, r) : x \in X,\, r > 0\}$, then the metric topology on X is

$$\mathcal{T}_d := \{A \subseteq \Omega : \forall\, x \in A,\, \exists\, B \in \mathcal{B} \text{ such that } x \in B \subseteq A\}.$$

Motivated by this, for any set X and for a collection $\mathcal{B}$ of subsets of X, let us consider

$$\mathcal{T}_{\mathcal{B}} := \{A \subseteq X : \forall\, x \in A,\, \exists\, B \in \mathcal{B} \text{ such that } x \in B \subseteq A\}.$$

Under what conditions on $\mathcal{B}$ can we assert that $\mathcal{T}_{\mathcal{B}}$ is a topology on X?

The following theorem gives an answer to the above question.

Theorem 2.55 *Let X be a set and $\mathcal{B}$ be a family of subsets of X such that $\mathcal{B}$ covers X and for every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. Then*

$$\mathcal{T} := \{A \subseteq X : \textit{for every } x \in A,\, \exists\, B \in \mathcal{B} \textit{ such that } x \in B \subseteq A\}.$$

is a topology on X and $\mathcal{B}$ is a (unique) base for $\mathcal{T}$.

Proof. It is enough to show that $\mathcal{T}$ is a topology on X. Clearly $\varnothing \in \mathcal{T}$, as $\varnothing$ contains no elements. Also, $X \in \mathcal{T}$, since $\mathcal{B}$ covers X. Now, let $A_1, A_2 \in \mathcal{T}$. To show that $A_1 \cap A_2 \in \mathcal{T}$. For this, let $x \in A_1 \cap A_2$. We know that there exists $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq A_1$ and $x \in B_2 \subseteq A_2$. Hence, $x \in B_1 \cap B_2 \subseteq A_1 \cap A_2$. By assumption, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Thus, $x \in B_3 \subseteq A_1 \cap A_2$. This shows that $A_1 \cap A_2 \in \mathcal{T}$. Next, let $\{A_\alpha : \alpha \in \Lambda\} \subseteq \widetilde{\mathcal{T}}$ for some $\Lambda \neq \varnothing$. To show that $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{T}$. For this, let $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Then there exists $\beta \in \Lambda$ such that $x \in A_\beta$, and hence, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq A_\beta \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. $\square$

Exercise 2.56 Deduce Theorem 2.54 from Theorem 2.55. ◁

Remark 2.57 We may observe that, in the context of a metric space (X, d), the conditions in Theorem 2.55 are satisfied: Note that

$$X = \bigcup_{x \in X} B(x, r)$$

for any $r > 0$. Now, consider open balls $B(x, r)$ and $B(y, s)$ for some $x, y \in X$ and $r, s > 0$. Let $u \in B(x, r) \cap B(y, s)$. Then, for small enough $t > 0$, we have $B(u, t) \subseteq B(x, r) \cap B(y, s)$. For example, we may take

$$0 < t < \min\{d(x, u), r - d(x, u), d(y, u), s - d(y, u)\}.$$

Thus, we have shown that the family

$$\mathcal{B} := \{B(x, r) : x \in X,\ r > 0\}$$

satisfies the conditions in Theorem 2.55. ♢

Definition 2.58 (Sub-base) Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{S}$ is a collection of subsets of X such that $\mathcal{T}$ is the topology generated by $\mathcal{S}$, then $\mathcal{S}$ is called a **sub-base** for $\mathcal{T}$. ♢

- Every base is a sub-base, but the converse is not true.

Example 2.59 Let $\mathcal{S}$ be the collection of all intervals of the form (a, ∞) and $(-\infty, b)$. Then $\mathcal{S}$ is a sub-base for the usual topology $\mathcal{T}_u$ on $\mathbb{R}$, but $\mathcal{S}$ is not a base for $\mathcal{T}_u$. For instance, $(-1, 1)$ is an open set containing 0. But, there is no set in $\mathcal{S}$ containing 0 and contained in $(-1, 1)$. ♢

Theorem 2.60 *Let X be a nonempty set and $\mathcal{S}$ be a family of subsets of X. Let $\mathcal{T}$ be the collection of all unions of finite intersections of members of $\mathcal{S}$, together with the sets $\varnothing$ and X. Then $\mathcal{T}$ is a topology on X, and it is the topology generated by $\mathcal{S}$.*

Proof. Clearly, $\mathcal{T}$ is a topology on X. Also, if $\widetilde{\mathcal{T}}$ is any topology containing $\mathcal{S}$, then $\widetilde{\mathcal{T}}$ must contain $\mathcal{T}$. □

Theorem 2.61 *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{S}$ be a family of subsets of X, and $\mathcal{B}$ be the family of all finite intersections of members of $\mathcal{S}$. Then $\mathcal{S}$ is a sub-base for $\mathcal{T}$ iff $\mathcal{B}$ is base for $\mathcal{T}$.*

Proof. Suppose $\mathcal{S}$ is a sub-base for $\mathcal{T}$. Then, by Theorem 2.60 and Theorem 2.49, $\mathcal{B}$ is a base for $\mathcal{T}$. Conversely, suppose $\mathcal{B}$ is a base for $\mathcal{T}$. Then, by Theorem 2.49, $\mathcal{S}$ is a subs-base for $\mathcal{T}$. □

Chapter 3

Countability Axioms and Separability

We have many examples of topological spaces and bases for them. Also, we know that the base can be a much smaller sub-collection than the topology itself. For example, for a discrete space on a countable set, the collection of all singleton sets is a base. In this case, the topology can be an uncountable collection, whereas a base can be a countable collection.We may also observe that if a base for a topology contains only a finite number of members, then the topology itself contains a finite number of members. Thus, if a topology consists of an infinite number of sets, then every base of such a topology must contain an infinite number of sets.This chapter explores what happens when these infinite bases are restricted to being countable. We will study spaces that possess a countable base at each point (First Countable) or a countable global base (Second Countable). Finally, we will examine how these countability restrictions allow a space to contain a dense, countable subset—a property known as separability.

3.1 Second countable and separability

Definition 3.1 (Second countable space) A topological space or a topology is said to be **second countable** if it has a countable base. $\Diamond$

- If a topological space X is second countable, then X is said to satisfy the **second axiom of countability**.

  We shall define the notion of a first countable space soon.

Example 3.2 The space $\mathbb{R}^k$ with usual topology is second countable. This follows from the above theorem since the countable set $D := \mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$. In fact, the family $\{B(u, r) : u \in D,\, r \in \mathbb{Q}^+\}$ is a countable base for $\mathbb{R}^k$ with usual topology. $\Diamond$

Example 3.3 A discrete space is second countable iff it is a countable set. $\Diamond$

For the next theorem, recall that a subset D of a topological space X is dense in X if $\bar{D} = X$, that is, for every $x \in X$, there exists an open neighbourhood G of X such that $G \cap D \neq \varnothing$.

Theorem 3.4 *Every metric space with a countable dense subset is second countable.*

Proof. Let X be a metric space with a metric d and let D be a countable dense subset of X. Let (ε_n) be a sequence of positive reals such that $\varepsilon_n \to 0$. Consider the family

$$\mathcal{A} := \{B(u, \varepsilon_n) : u \in D, n \in \mathbb{N}\}.$$

Clearly, $\mathcal{A}$ is a countable family of open sets. We show that $\mathcal{A}$ is a base for X. For this, let A be an open set in X and $x \in A$. Let $r > 0$ be such that $B(x, r) \subseteq A$. Let $n \in \mathbb{N}$ be large enough such that $\varepsilon_n < r/2$. Since D is dense in X, the ball $B(x, \varepsilon_n)$ contains some point $u \in D$. Then

$$x \in B(u, \varepsilon_n) \subseteq B(x, r) \subseteq A.$$

Indeed, $u \in B(x, \varepsilon_n)$ implies $x \in B(u, \varepsilon_n)$ and if $v \in B(u, \varepsilon_n)$, then

$$d(v, x) \leq d(v, u) + d(u, x) < \varepsilon_n + \varepsilon_n < r$$

so that $B(u, \varepsilon_n) \subseteq B(x, r)$ as well. Thus, we have shown that for every $x \in A$, there exists $B \in \mathcal{A}$ such that $x \in B \subseteq A$, so that $\mathcal{A}$ is a countable base for X. $\square$

Definition 3.5 (Separable space) A topological space or a topology is said to be **separable** if it has a countable dense subset. $\Diamond$

Example 3.6 The metric space $X = \mathbb{R}^k$ with usual metric is separable. For example, the set $\mathbb{Q}^k$ is a countable dense subset of X. $\Diamond$

Example 3.7 Every indiscrete topological space X separable and second countable. In fact, the only base of the indiscrete space X in $\{X\}$, and every nonempty subset of X is dense in X, in particular, every countable subset of X is dense in X. $\Diamond$

Every separable metric space is second countable (Theorem 3.4). An uncountable discrete space is neither separable nor second countable.

Theorem 3.8 *Every second countable space is separable.*

Proof. Let $(X, \mathcal{T})$ be a second countable space and let $\mathcal{B} = \{B_n : \alpha \in \Lambda\}$ be a countable base for $\mathcal{T}$, where Λ is some subset of $\mathbb{N}$. For each $n \in \Lambda$, let $x_n \in B_n$ and let $D := \{x_n : n \in \Lambda\}$. Clearly, D is a countable set. We claim that D is dense in X. For this, let $x \in X$ and let G be an open neighbourhood of x. Since $\mathcal{B}$ is a base for X, there exists $k \in \Lambda$ such that $x \in B_k \subseteq G$. In particular, $x_k \in B_k \cap G$, so that $D \cap G \neq \varnothing$. Hence, D is dense in X. $\square$

We have proved that (see Theorem 3.4) every separable metric space is second countbale.

- Is every separable topological space second countable? Not necessarily, as the following theorem shows.

Theorem 3.9 *The lower limit topology and the upper limit topology on* $\mathbb{R}$ *are separable, but not second countable. In particular, they are not metrizable.*

Proof. Let $\mathbb{R}_\ell := \mathbb{R}$ with lower limit topology.

(a) It can be easily seen that $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$, and hence it is separable (Exercise).

(b) Let $\mathcal{B}$ be a base for $\mathbb{R}_\ell$. Then for each $x \in \mathbb{R}_\ell$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, \infty)$. Clearly, $\inf B_x = x$. Hence, $x \neq y$ implies $B_x \neq B_y$. Hence, $\{B_x : x \in \mathbb{R}_\ell\}$ is uncountable, and therefore, $\mathcal{B}$ is uncountable. Thus, $\mathbb{R}_\ell$ is not second countable.

(c) Since $\mathbb{R}_\ell$ is separable and not second countable, by Theorem 3.4, it is not metrizable. $\square$

We know that, with respect to the usual metric, $\mathbb{R}$ is separable. Also, as a subspace of $\mathbb{R}$, $\mathbb{Q}$ is separable. Is $\mathbb{R} \setminus \mathbb{Q}$ separable? The answer is in the affirmative, which follows from the following general result.

Theorem 3.10 *Every subspace of a separable metric space is separable.*

Proof. Let (X, d) be a separable metric space and X_0 be a subset of X. We show that X_0 is also separable. Clearly, if X_0 is countable, then it is separable. So assume that X_0 is uncountable. Then D has to be countably infinite (Why?). Let $D = \{x_n : n \in \mathbb{N}\}$. Then, for each $n \in \mathbb{N}$, we have $X = \bigcup_{i=1}^{\infty} B(x_i, 1/n)$ so that

$$X_0 = \bigcup_{i=1}^{\infty} B(x_i, 1/n) \cap X_0 \quad \forall\, n \in \mathbb{N}.$$

Also, for every $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $B(x_j, 1/n) \cap X_0 \neq \varnothing$. Let

$$\Delta := \{(j,n) \in \mathbb{N} \times \mathbb{N} : B(x_j, 1/n) \cap X_0 \neq \varnothing\}.$$

For $(j,n) \in \Delta$, let $y_{j,n} \in B(x_j, 1/n) \cap X_0$. Let

$$D_0 := \{y_{j,n} : (j,n) \in \Delta\}.$$

We show that D_0 is dense in X_0. For this, let $x \in X_0$ and $\varepsilon > 0$ be given. We have to show that $B(x, \varepsilon) \cap D_0 \neq \varnothing$. Note that, for every $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $x \in B(x_j, 1/n)$ so that $(j,n) \in \Delta$. Let n be large enough such that $2/n < \varepsilon$. Then, we have

$$d(x, y_{j,n}) \leq d(x, x_j) + d(x_j, y_{j,n}) < \frac{1}{n} + \frac{1}{n} < \varepsilon.$$

Thus, $B(x, \varepsilon) \cap D_0 \neq \varnothing$, and the proof is complete. $\square$

At this point, one may ask: Is subspace of every separable topological space separable? The answer to this question is in the negative, as the following example shows.

Example 3.11 Let $\mathcal{T}$ consists of $\varnothing$ and those $A \subseteq \mathbb{R}^2$ such that for every $(x,y) \in A$, there exists $r > 0, s > 0$ such that

$$[x, x+r) \times [y, y+s) \in A.$$

It can be easily seen that $\mathcal{T}$ is a topology on $\mathbb{R}^2$. Let $X = \mathbb{R}^2$ with this topology $\mathcal{T}$. As in the case of lower limit topology, X is separable. We show that the set $X_0 = \{(x, -x) : x \in \mathbb{R}\}$ is not separable. To see this, we observe that, for every $x \in \mathbb{R}$,

$$\{(x, -x)\} = X_0 \cap ([x, x+r) \times [-x, -x+r))$$

for any $r > 0$. Hence, every singleton subset of X_0 is open in X_0, so that the subspace topology on X_0 is the discrete topology. Since an uncountable set with discrete topology is not separable, X_0 is not separable. $\Diamond$

3.2 First countability

Definition 3.12 Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A family $\mathcal{B}_x$ of open neighbourhoods of x is said to be a **base at** x or **local base at** x if every open set containing x contains some member of $\mathcal{B}_x$. $\Diamond$

Definition 3.13 (First countable space) A topological space X is said to be **first countable** if it has a countable local base at every point in X. $\Diamond$

- If a topological space X is first countable, then X is said to satisfy the **first axiom of countability**.

Example 3.14 (i) In $\mathbb{R}$ with usual topology, the set $\mathcal{B}_x := \{(x-1/n, x+1/n) : n \in \mathbb{N}\}$ is a local base at $x \in \mathbb{R}$.

(ii) If $\mathbb{R}$ is with lower limit topology, then $\mathcal{B}_x := \{[x, x+1/n) : n \in \mathbb{N}\}$ is a local base at $x \in \mathbb{R}$ for every $x \in \mathbb{R}$.

(ii) In a discrete space X, the family $\mathcal{B}_x = \{\{x\}\}$ is a local base at x.

(iii) In an indiscrete space X, $\mathcal{B}_x = \{X\}$ is the only local base at x. $\Diamond$

Theorem 3.15 *Every metric space is first countable.*

Proof. Let (Ω, d) be metric space. Then, for every $x \in \Omega$, the family $\mathcal{B}_x := \{B_d(x, 1/n) : n \in \mathbb{N}\}$ is a countable base at x: For every open set A containing x, there exists $r > 0$ such that $B_d(x, r) \subseteq A$. Taking $n \in \mathbb{N}$ such that $1/n < r$, we have $B_d(x, 1/n) \subseteq B_d(x, r) \subseteq A$. $\square$

Theorem 3.16 *Every second countable space is first countable, and the converse is not true.*

Proof. Let $(X, \mathcal{T})$ be a second countable space, and let $\mathcal{B}$ be a countable base for X. Let $x \in X$, and let

$$\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\}.$$

Then $\mathcal{B}_x$ is a countable base at x. Indeed, for every open set A containing x, there exists $B_x \in \mathcal{B}$ such that $x \in B \subseteq A$. Clearly $B_x \in \mathcal{B}_x$. This shows that $\mathcal{B}_x$ is a countable base at x.

The converse of the above theorem is not true. To see this, it is enough consider an uncountable discrete space X. This space is not second countable. However, for each $x \in X$, the family $\mathcal{B}_x := \{\{x\}\}$ is a base at x. $\square$

Every metric space is first countable, but not necessarily separable and second countable.

3.3 First countability and convergence

By the definition of closure of a set, we obtain the following theorem.

Theorem 3.17 *Let A be a subset of a topological space X. If (x_n) is a sequence in A such that $x_n \to x$, then $x \in \bar{A}$.*

We know that, if X is a metric space, then the converse of the above theorem is also true. However, in a general topological space, the converse need not be true, as the following example shows.

Example 3.18 Let $X = \mathbb{R}$ with co-countable topology and $A = \mathbb{R} \setminus \mathbb{Q}$. Then $\bar{A} = X$, since for every $x \in \mathbb{R}$ every open neighbourhood G of x is an uncountable set. However, if $x \in \mathbb{Q}$, then there is no sequence in A which converges to x, as every convergent sequence in X is eventually constant.

More generally, let X be any uncountable set with co-countable topology and let $A = X \setminus B$, where B is a nonempty countable set. Then, following the arguments as in last paragraph, we see that $\bar{A} = X$ and a point in B is not a limit of any sequence in A. $\Diamond$

However, if X is a first countable space, then converse to Theorem 3.17 does hold.

Theorem 3.19 *Suppose X is a first countable space and $A \subseteq X$. Then $x \in \bar{A}$ iff there exists a sequence (x_n) in A such that $x_n \to x$.*

Proof. Let $x \in \bar{A}$. Let $\mathcal{B}_x := \{B_n : n \in \mathbb{N}\}$ be a countable base at x. Let $G_n := \bigcap_{j=1}^n B_j$. Since $x \in \bar{A}$ and G_n is an open set containing x, there exists $x_n \in G_n \cap A$ for all $n \in \mathbb{N}$. We claim that $x_n \to x$. To see this, let G be an open set containing x. Since $\mathcal{B}_x$ is a countable base at x, there exists $k \in \mathbb{N}$ such that $B_k \subseteq G$. Note that

$$G_n \cap A \subseteq B_k \subseteq G \quad \forall n \geq k.$$

In particular, $x_n \in G$ for all $n \geq k$. Thus, we have proved that $x_n \to x$.

Conversely, suppose (x_n) is a sequence in A such that $x_n \to x$. Let G be any open set containing x. Then, by the definition of convergence, there exists $k \in \mathbb{N}$ such that $x_k \in G$. In particular, $G \cap A \neq \varnothing$, showing that $x \in \bar{A}$. $\square$

An uncountable co-countable topological space is not first countable. (see Theorem 3.19 and Example 3.18.)

- With respect to co-countable topology $\mathbb{R}$, $\mathbb{Q}$ is a proper closed subset of $\mathbb{R}$, whereas $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Recall that:

- In a Hausdorff topological space, every convergent sequence has a unique limit.
- Is the converse true? The answer is in the negative, as the following example shows.

Example 3.20 Consider the co-countable topology on an uncountable set X. We know that this space is not Hausdorff, and every convergent sequence is eventually constant, and hence has a unique limit (cf. Theorem 2.13). $\Diamond$

However, we have the following theorem.

Theorem 3.21 *Suppose X is a first countable topological space. Then every convergent sequence has a unique limit iff X is Hausdorff.*

Proof. We have already proved that if X is Hausdorff, then every convergent sequence in X has a unique limit.

Conversely, suppose X is first countable and every convergent sequence in X has a unique limit. To show that X is Hausdorff. For a moment assume that X is not Hausdorff. Then there exists $u, v \in X$ with $u \neq v$ such that every pair of open sets containing u and v, respectively, will intersect. Now, let $\mathcal{B}_u = \{U_n : n \in \mathbb{N}\}$ and $\mathcal{B}_v = \{V_n : n \in \mathbb{N}\}$ be countable bases at u and v, respectively. Let

$$A_n = \bigcap_{j=1}^{n} U_j, \quad B_n = \bigcap_{j=1}^{n} V_j \quad \forall\, n \in \mathbb{N}.$$

Then A_n and B_n are open sets containing u and v, respectively, and hence $A_n \cap B_n \neq \varnothing$. Let $x_n \in A_n \cap B_n$. We show that $x_n \to u$ and $x_n \to v$, which would contradict the hypothesis. For this, let G be an open set containing u. Since $\mathcal{B}_u$ is a base at u, there exists $k \in \mathbb{N}$ such that $U_k \subseteq G$. Hence $A_n \subseteq G$ for all $n \geq k$, so that we also have

$$x_n \in A_n \cap B_n \subseteq A_n \subseteq G \quad \forall\, n \geq k.$$

Thus, $x_n \to u$. Analogously, using the fact that $\mathcal{B}_v$ is a local base at v, we see that $x_n \to v$. This completes the proof. $\Box$

- The co-countable topology on an uncountable space is not first countable, and hence, not second countable as well (See Example 3.20 and Theorem 3.21).

Chapter 4

Functions and Their Properties

The subject of calculus deals extensively with the properties of functions between subsets of $\mathbb{R}$ and higher-dimensional Euclidean spaces. The same is true for advanced analysis involving metric spaces. In this chapter, we shall deal with analogous cases in the setting of topological spaces. We will see that certain results which hold true in the context of metric spaces need not be true in the context of general topological spaces. However, generalizing our study to topology is essential to include these broader, more abstract situations.

4.1 Continuity and sequential continuity

Motivated by the characterization of continuity of a function at a point in the context of metric spaces, we have the following definition in the context of topological spaces.

Definition 4.1 Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous at a point** $x \in X$ if for every open neighbourhood V of $f(x)$, there exists an open neighbourhood U of x such that $f(U) \subseteq V$. $\Diamond$

We also define the following.

Definition 4.2 Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous on a subset** E of X if it is continuous at every point in E, and f is said to be **continuous** if it is continuous on X. $\Diamond$

Exercise 4.3

1. If X is a discrete space and Y is any topological space, then every functions $f : X \to Y$ is continuous.
2. Let X be a nonempty set and $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Let $X_1 = X$ with $\mathcal{T}_1$ and $X_2 = X$ with $\mathcal{T}_2$. Then the identity map from X_1 to X_2 is continuous iff $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$. $\Diamond$

Theorem 4.4 *Let $f : X \to Y$ be a function between topological spaces X and Y. Then f is continuous iff for every open set G in Y, $f^{-1}(G)$ is open in X.*

Proof. Suppose f is continuous. Let G be an open set in Y. To show that $f^{-1}(G)$ is open in X. For this, let $x \in f^{-1}(G)$. Then $f(x) \in G$. Hence, using the continuity of f at x, there exists an open set U in X such that $x \in U$ and $f(U) \subseteq G$. This implies that $U \subseteq f^{-1}(G)$. Thus, we have proved that for every $x \in f^{-1}(G)$, there exists an open set U in X such that $x \in U \subseteq f^{-1}(G)$. Hence, $f^{-1}(G)$ is open.

Conversely, assume that for every open set G in Y, $f^{-1}(G)$ is open in X. We have to show that f is continuous on X. So, let $x \in X$ and V be an open neighbourhood of $f(x)$ in Y. By assumption, $U := f^{-1}(V)$ is an open set in X. Since $f(x) \in V$, U is an open neighbourhood of x in X. Note that

$$y \in U \quad \Longrightarrow \quad f(y) \in V, \quad \textit{i.e.}, \quad f(U) \subseteq V.$$

This shows that f is continuous. $\square$

Remark 4.5 In some of the books on topology, the continuity of a function is defined in terms of the above characterization. $\diamondsuit$

Theorem 4.6 *Let X and Y be topological spaces. Then $f : X \to Y$ is continuous iff for every closed set B in Y, $f^{-1}(B)$ is closed in X.*

Proof. Follows by observing that for $B \subseteq Y$, $f^{-1}(B^c) = (f^{-1}(B))^c$. $\square$

The proof of the following theorem follows using the definitions of base and sub-base for a topology.

Theorem 4.7 *Let X and Y be topological spaces, $\mathcal{B}$ be a base and $\mathcal{S}$ be a sub-base for the topology on Y. The, for a function $f : X \to Y$, the following are equivalent:*

(i) *f is continuous.*

(ii) *$f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.*

(iii) *$f^{-1}(S)$ is open in X for every $S \in \mathcal{S}$.*

Exercise 4.8 Supply details of the proof for the above theorem. $\lhd$

Example 4.9 Let X be any topological space and $\mathbb{R}$ be with usual topology. Then a function $f : X \to \mathbb{R}$ is continuous iff

$$\{x \in X : a < f(x) < b\}$$

is open in X for any $a, b \in \mathbb{R}$ with $a < b$. $\diamondsuit$

Theorem 4.10 *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function. Then, for any sequence (x_n) in X,*

$$x_n \to x \text{ in } X \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } Y.$$

Proof. Let (x_n) be a sequence in X such that $x_n \to x$ in X. To show that $f(x_n) \to f(x)$. For this, let A be an open set containing $f(x)$. Since f is continuous, $f^{-1}(A)$ is open in X. Also, $x \in f^{-1}(A)$. Since $x_n \to x$ and $f^{-1}(A)$ is an open set containing x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in f^{-1}(A)$ for all $n \geq n_0$. This implies that $f(x_n) \in A$ for all $n \geq n_0$. Thus, $f(x_n) \to f(x)$. $\square$

Remark 4.11 By the above theorem. if we can find a sequence (x_n) in X such that $x_n \to x$ for some $x \in X$ and $f(x_n) \not\to f(x)$, then f is not continuous. $\Diamond$

- Converse of Theorem 4.10 is not true. To see this, look at the following example.

Example 4.12 Let $\mathbb{R}_{cc} = \mathbb{R}$ with co-countable topology and $\mathbb{R}_{dis} = \mathbb{R}$ with discrete topology. Consider the identity function $f : \mathbb{R}_{cc} \to \mathbb{R}_{dis}$, that is, $f(x) = x$ for every $x \in \mathbb{R}$. This function is not continuous. For example, $\{1\}$ is an open set in $\mathbb{R}_{dis}$, but $f^{-1}(\{1\})$ is not open, as its compliment is not countable. However, for every sequence (x_n) in $\mathbb{R}$,

$$x_n \to x \text{ in } \mathbb{R}_{cc} \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } \mathbb{R}_{dis}.$$

This is true, because, if $x_n \to x$ in $\mathbb{R}_{cc}$, then (x_n) is eventually constant, and hence $x_n \to x$ in $\mathbb{R}_{dis}$. $\Diamond$

More generally we have the following example.

Example 4.13 Let X be a co-countable topological space and Y be any topological space. Let $f : X \to Y$ be any function. We know that if (x_n) is a convergent sequence in X, then it is eventually constant, and hence its image $(f(x_n))$ is also eventually constant. In other words,

$$\begin{aligned} x_n \to x &\Longrightarrow \exists\, N \in \mathbb{N} \text{ such that } x_n = x \quad \forall\, n \geq N \\ &\Longrightarrow f(x_n) = f(x) \quad \forall\, n \geq N \\ &\Longrightarrow f(x_n) \to f(x). \end{aligned}$$

However, this f need not be continuous. For example, if X is uncountable co-countable space, Y is with discrete topology and if f is one-one, then f is not continuous. Because, for $y \in Y$, the singleton set $\{y\}$ is open in Y, but $f^{-1}(\{y\})$, which is a singleton set, is not open in X. $\Diamond$

Definition 4.14 (Sequential continuity) Let X and Y be topological spaces. Then a function is said to be **sequentially continuous** if for any sequence (x_n) in X,

$$x_n \to x \text{ in } X \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } Y. \qquad \Diamond$$

- Continuity implies sequential continuity.

> If X is an uncountable set with co-countable topology, then every function from X to any topological space Y is sequentially continuous.

In certain special cases, continuity and sequential continuity are equivalent.

Theorem 4.15 *Let X and Y be topological spaces. If X is a metrizable space, then continuity and sequential continuity of a function $f : X \to Y$ are equivalent.*

Proof. We know that if $f : X \to Y$ is continuous, then f is sequentially continuous. Now, suppose $f : X \to Y$ is sequentially continuous, and X is metrizable. In this case we show that f is continuous.

Let d be a metric which induces the topology on X, and assume for a moment that f is not continuous. Then there exists $x_0 \in X$ and an neighbourhood V of $f(x_0)$ such that for any open neighbourhood U of x_0, there exists $x \in U$ such that $f(x) \notin V$. In particular, for any $n \in \mathbb{N}$, taking U as $B_X(x_0, 1/n)$, there exists $x_n \in B_X(x_0, 1/n)$ such that $f(x_n) \notin V$ for all $n \in \mathbb{N}$. Thus, we obtain sequence (x_n) in X such that $x_n \to x_0$, but $f(x_n) \not\to f(x_0)$. This is a contradiction to the fact that f is sequentially continuous. $\Box$

> Continuity and sequential continuity are equivalent if the domain space is metrizable.

More generally, we show that, if X is first countable, then continuity of a function $f : X \to Y$ is equivalent to sequentially continuous. For proving this, we make use of the following characterization of continuity in a general setting.

Theorem 4.16 *Let X and Y be topological spaces. Then $f : X \to Y$ is continuous iff for every $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.*

Proof. $\Rightarrow$): Suppose f is continuous. Let $x \in \bar{A}$. We have to show that $f(x) \in \overline{f(A)}$. Let V be an open neighbourhood of $f(x)$. To show that V contains some point from $f(A)$. Since f is continuous, we know that $f^{-1}(V)$ is an open neighbourhood of x, and since $x \in \bar{A}$, the open neighbourhood $f^{-1}(V)$ of x contains some point from A, say $u \in f^{-1}(V) \cap A$. Then $f(u) \in V \cap f(A)$.

$\Leftarrow$): Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$. We have to show that f is continuous. By Theorem 4.6, it is enough to show that $f^{-1}(B)$ closed for every closed $B \subseteq Y$. That is, taking $A = f^{-1}(B)$, we have to show that $A = \bar{A}$. So, let $x \in \bar{A}$. Then, by hypothesis, $f(x) \in \overline{f(A)}$. But, since $A = f^{-1}(B)$, we have $f(A) \subseteq B$, so that

$$f(x) \in \overline{f(A)} \subseteq \bar{B} = B.$$

Thus, we have proved that if $x \in \bar{A}$, then $x \in f^{-1}(B) = A$, showing that $\bar{A} = A$. $\square$

Let us recapitulate some of the results which we have already proved: Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent.

(i) f is continuous.

(ii) For every open set V in Y, $f^{-1}(V)$ is open in X.

(iii) For every closed set F in Y, $f^{-1}(F)$ is closed in X.

(iv) For every set $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

Now, we prove the promised theorem.

Theorem 4.17 *Let X and Y be topological spaces. If X is first countable, then a function $f : X \to Y$ is continuous iff it is sequentially continuous.*

Proof. Let X and Y be topological spaces and X be a first countable space. Suppose $f : X \to Y$ is sequentially continuous. We have to show that f is continuous. By Theorem 4.16, it is enough to show that $f(\bar{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$. So, let $A \subseteq X$ and $x \in \bar{A}$. Since X is first countable, by Theorem 3.19, there exists (x_n) in A such that $x_n \to x$. By assumption $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, it follows that $f(x) \in \overline{f(A)}$, and the proof is complete. $\square$

- Continuity and sequential continuity are equivalent if the domain space is first countable.

4.2 Continuity and separability

Lemma 4.18 *Continuous image of a dense subset is dense in the range space.*

Proof. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. Let D be a dense subset of X. We show that $f(D)$ is dense in $f(X)$. For this, let $y \in f(X)$ and V be an open set in $f(X)$ which contains y. Then $V = G \cap f(X)$, where G is open in Y and $y \in G$. We have to show that $f(D) \cap V \neq \varnothing$.

Let $x \in X$ be such that $y = f(x)$. Since f is continuous, $f^{-1}(G)$ is an open neighbourhood of x. Since D is dense in X, $D \cap f^{-1}(G) \neq \varnothing$. Let $u \in D \cap f^{-1}(G)$. Clearly, $f(u) \in f(D) \cap G \subseteq f(D) \cap V$. Thus, $f(D) \cap V \neq \varnothing$ and the proof is complete. $\square$

As a corollary to the above lemma, we have the following theorem.

Theorem 4.19 *Continuous image of a separable space is separable in the range space.*

Proof. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. Suppose X is separable and D be a countable dense subset of X. By Lemma 4.18, $f(D)$ is dense in X. Clearly, $f(D)$ is a countable subset of $f(X)$. $\square$

Example 4.20 Let $\mathcal{P}_k$ be the set of all polynomials of degree atmost k with real coefficients and let $\mathcal{P}$ be the set of all polynomials with real coefficients. Let

$$d(f, g) = \sup_{0 \leq t \leq 1} |f(t) - g(t)| \quad \text{for} \quad f, g \in \mathcal{P}.$$

It can be easily seen that $d(\cdot, \cdot)$ is a metric on $\mathcal{P}$. We show that

- $X_k := \mathcal{P}_k$ and $X := \mathcal{P}$ are separable with respect to the metric $d(\cdot, \cdot)$ above.

For thus, let $\mathbb{R}^{k+1}$ be with the metric

$$\rho(u, v) := |u_1 - v_1| + \cdots + |u_k - v_k|.$$

consider the function $\varphi_k : \mathbb{R}^{k+1} \to X_k$ defined by

$$\varphi(a_0, a_1, \ldots, a_k) = a_0 + a_1 t + \cdots + a_k t^k.$$

Note that φ is a bijective function. I can be easily seen, using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, that $\mathbb{Q}^{k+1}$ is dense in $\mathbb{R}^{k+1}$ (Verify). Note that

$$|\varphi(a_0, a_1, \ldots, a_k)| \leq |a_0| + |a_1| + \cdots + |a_k|.$$

From this, it follows that

$$d(\varphi(u), \varphi(v)) = |\varphi(u) - \varphi(v)| \leq \rho(u, v).$$

Hence, φ is a continuous function. Hence, by Theorem 4.19 that $\varphi(\mathbb{Q}^{k+1})$ is a countable dense subset of $\mathcal{P}_k$.

Since , $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$, X is also separable. $\Diamond$

Exercise 4.21 Subspace of a separable space is separable. $\triangleleft$

4.3 Restriction, expansion and composition theorems

The proofs of all the items in the following theorem follow from the definition of continuity. However, we supply brief explanations.

Theorem 4.22 *Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. Then the following results hold.*

(1) **(Domain restriction theorem)** *Let X_0 be a subspace of X. Then the restriction of f, namely, the function $f_0 : X_0 \to Y$ defined by*

$$f_0(x) = f(x) \quad \forall\, x \in X,$$

is continuous.

(2) **(Co-domain restriction theorem)** *Let Y_0 be a subspace of Y with $f(X) \subseteq Y_0$. Then the function $g : X \to Y_0$ defined by*

$$g(x) = f(x) \quad \forall\, x \in X,$$

is continuous.

(3) **(Co-domain expansion theorem)** *Let Z be a topological space such that Y is a subspace of Z. Then the function $h : X \to Z$ defined by*

$$h(x) = f(x) \quad \forall\, x \in X,$$

is continuous.

Proof. (1) Let A be any subset of Y. Then we see that

$$f_0^{-1}(A) = f^{-1}(A) \cap X_0.$$

In particular, for every open set V in Y, $f_0^{-1}(V) = f^{-1}(V) \cap X_0$ is open in X_0.

(2) Let V be open in Y_0. Then $V = G \cap Y_0$, where G is open in Y. Then, we have

$$g^{-1}(V) = f^{-1}(V) = f^{-1}(G) \cap X = f^{-1}(G),$$

which is open in X.

(3) Let G be open in Z. Then $V = G \cap Y$ is open in Y and we have

$$f^{-1}(V) = f^{-1}(G \cap Y).$$

But, for $x \in X$,

$$x \in f^{-1}(G \cap Y) \iff f(x) \in G \cap Y \iff f(x) \in G \iff h(x) \in G.$$

Thus, $f^{-1}(V) = h^{-1}(G)$,where $f^{-1}(V)$ is open in X. Thus, inverse image of every open set in G under h is open in X, showing that h is continuous. $\square$

Theorem 4.23 (Composition theorem) *Let X, Y and Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then the composition function $g \circ f : X \to Z$ is continuous.*

Proof. Let V be an open set in Z. Then, we have

$$(g \circ f)^{-1}(V) = f^{-1}[g^{-1}(V)],$$

which is open in X. $\square$

Definition 4.24 Given a function $f : X \to Y$ and a subset X_0 of X, the function $f_0 : X_0 \to Y$ defined by

$$f_0(x) = f(x), \quad x \in X_0,$$

is called the **restriction** of f to X_0, and it is usually denoted by $f|_{X_0}$, and in that case, f is called an **extension** of f_0. $\Diamond$

- Restriction of every continuous function is continuous. But, the converse need not be true, as the following example shows.

Example 4.25 Let $X = \mathbb{R}$ be with usual topology and $X_0 = [0, 1]$ be with the subspace topology. Then $f_0 : X_0 \to X$ defined by

$$f_0(x) = x, \quad x \in X_0,$$

is continuous. Let $f : X \to X$ be define by

$$f(x) = \begin{cases} x, & x \in X_0, \\ 0, & x \in X \setminus X_0. \end{cases}$$

We note that f is an extension of f_0, and f is not continuous. ♢

- Does every continuous function defined on a subspace has a continuous extension to all of X? Not necessarily, as the following example shows.

Example 4.26 Let $\mathbb{R}$ be with usual topology and $((0, \infty)$ is with subspace topology. Let $f_0 : (0, \infty) \to \mathbb{R}$ be defined by

$$f_0(x) = \frac{1}{x}, \quad x \in (0, \infty).$$

Then f_0 is continuous. But, it does not have a continuous extension to $\mathbb{R}$ (Why?). ♢

In the following theorem[1], we give a condition on the subspace which guarantees continuous extension.

Theorem 4.27 *Let X, Y be topological spaces, X_0 be a subspace of X and $f_0 : X_0 \to Y$ be continuous. If X_0 is open as well as closed, then f_0 has a continuous extension to all of X.*

Proof. Let $f_1 : X_0^c \to Y$ be any continuous function (e.g., we may take $f_1(x) = y_0$ for every $x \in X_0^c$ for any given $y_0 \in Y$). Then define $f : X \to Y$ by

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in X_0, \\ f_1(x) & \text{if } x \in X_0^c, \end{cases}$$

Now, let V be an open set in Y. Then we have

$$f_0^{-1}(V) = f^{-1}(V) \cap X_0, \quad f_1^{-1}(V) = f^{-1}(V) \cap X_0^c.$$

Hence,

$$f^{-1}(V) = f_0^{-1}(V) \cup f_1^{-1}(V).$$

Since f_0 and f_1 are continuous, the sets $f_0^{-1}(V)$ and $f_1^{-1}(V)$ are open in X_0 and X_0^c, respectively. Now, since X_0 and X_0^c are open in X, the sets $f_0^{-1}(V)$ and $f_1^{-1}(V)$ are open in X as well. Thus, $f^{-1}(V)$ is open in X. Thus, we have proved that f is a continuous extension of f_0. □

[1] This theorem is not explicitly stated in the referred books.

Exercise 4.28 Does $\mathbb{R}$ with usual topology have a nonempty proper subset which is open and closed? Why? ◁

Theorem 4.27 is a particular case of the following theorem, called the *pasting lemma.*

Theorem 4.29 (Pasting lemma) *Let X and Y be topological spaces, and A and B be closed subspaces of X such that $X = A \cup B$. If $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous functions such that*

$$f_1(x) = f_2(x) \quad \forall\, x \in A \cap B,$$

then the function $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B, \end{cases}$$

is a continuous function.

Proof. We observe that for every subset S of Y,

$$f^{-1}(S) = f_1^{-1}(S) \cup f_2^{-1}(S).$$

Now, if S is closed in Y, then $f_1^{-1}(S)$ closed in A and $f_2^{-1}(S)$ closed in B, and since A and B are closed, $f_1^{-1}(S)$ and $f_2^{-1}(S)$ are closed in X as well. Thus, we have proved that inverse every closed set under f is closed in X. □

Exercise 4.30 Prove the pasting lemma (Theorem 4.29) by assuming that both A and B are closed, instead of assuming that both are open. ◁

Continuous extension of a function is also possible under certain other conditions on the spaces and the function (see Appendix below).

4.4 Continuous extension in the context of metric spaces

We show that continuous extension of a function is possible in the context of metric spaces under certain conditions on the spaces and the function. First, let us recall the definition of *uniform continuity* of a function between metric spaces.

Definition 4.31 Let (X, d) and (Y, ρ) be metric spaces. Then a function $\varphi : X \to Y$ is **uniformly continuous** on a subset S of X if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S, \quad d(x, y) < \delta \quad \Longrightarrow \quad \rho(f(x), f(y)).$$

◊

Theorem 4.32 *Let (X, d) and (Y, ρ) be metric spaces, and X_0 be a dense subspace of X. If Y is is complete, then every uniformly continuous function $f_0 : X_0 \to Y$ has a unique continuous extension $f : X \to Y$.*

Proof. Suppose Y is complete and $f_0 : X_0 \to Y$ is a uniformly continuous function. Let $x \in X$. Since X_0 is dense in X, there exists a sequence (x_n) in X_0 such that $x_n \to x$. Since f_0 is uniformly continuous, it can be seen (Exercise) that the sequence $(f_0(x_n))$ is a Cauchy sequence in Y. Since Y is complete, $(f_0(x_n))$ converges. Let

$$f(x) = \lim_{n\to\infty} f_0(x_n). \tag{$*$}$$

We show that f is a well-defined continuous function from X to Y which is the unique extension of f_0.

(i) (Well defined extension): We show that f prescribed as in $(*)$ is a well-defined function from X to Y which is an extension of f_0: To show this we have to show that $f(x)$ independent of the sequence (x_n) which converges to x. For this, let (u_n) be any other sequence in X_0 which converges to x. Then the sequence $(f_0(u_n))$ is a Cauchy sequence in Yso that $(f_0(u_n))$ also converges. We show that $f(u_n) \to f(x)$. Since

$$\begin{aligned} \rho(f(u_n), f(x)) &\leq \rho(f(u_n), f(x_n)) + \rho(f(x_n), f(x)) \\ &\leq \rho(f_0(u_n), f_0(x_n)) + \rho(f_0(x_n), f(x)) \quad \forall\, n \in \mathbb{N} \end{aligned}$$

and $\rho(f_0(x_n), f(x)) \to 0$, it is enough to show that $\rho(f_0(u_n), f_0(x_n)) \to 0$.

Now, let $\varepsilon > 0$ be given. Since f_0 is uniformly continuous, there exists $\delta > 0$ such that

$$d(v, w) < \delta \quad \Longrightarrow \quad \rho(f(v), f(w)) < \varepsilon. \tag{1}$$

Since $d(x_n, x) \to 0$ and $d(u_n, x) \to 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \delta/2 \quad \text{and} \quad d(u_n, x) < \delta/2 \quad \forall\, n \geq N$$

so that

$$d(x_n, u_n) \leq d(x_n, x) + d(x, u_n) < \delta \quad \forall\, n \geq N. \tag{2}$$

Hence, by (1) and (2) above,

$$\rho(f(x_n), f(u_n)) < \varepsilon \quad \forall\, n \geq N.$$

Thus, we have shown that $\rho(f(u_n), f(x)) \to 0$, and the the value $f(x)$ is is uniquely defined, making $f : X \to Y$ a function.

Note that

$$f(x) = f_0(x) \quad \forall\, x \in X_0.$$

Thus, f is an extension of f_0.

(ii) (Continuity of extension): We show that f is continuous. For this, let $x \in X$ and $\varepsilon > 0$ be given. Since f_0 is uniformly continuous, there exists $\delta > 0$ such that

$$v, w \in X_0,\ d(v,w) < \delta \quad \Longrightarrow \quad \rho(f_0(v), f_0(w)) < \varepsilon. \tag{1}$$

For a $\delta_0 > 0$, let $u \in X$ be such that $d(x,u) < \delta_0$, and let (x_n) and (u_n) in X_0 such that $d(x_n, x) \to 0$ and $d(u_n, u) \to 0$. We note that

$$\rho(f(x), f(u)) \le \rho(f(x), f_0(x_n)) + \rho(f_0(x_n), f_0(u_n)) + \rho(f_0(u_n), f(u)) \tag{2}$$

for all $n \in \mathbb{N}$. By the definition of $f(x)$ and $f(u)$, we know that $\rho(f(x), f_0(x_n)) \to 0$ and $\rho(f(u_n), f_0(u)) \to 0$. Hence, there exists $N_1 \in \mathbb{N}$ such that

$$\rho(f(x), f_0(x_n)) < \varepsilon, \quad \rho(f(u_n), f_0(u)) < \varepsilon \quad \forall\, n \ge N_1. \tag{3}$$

Note that

$$d(x_n, u_n) \le d(x_n, x) + d(x, u) + d(u, u_n) \quad \forall\, n \in \mathbb{N}.$$

Since $d(x_n, x) \to 0$ and $d(u, u_n) \to 0$, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\delta}{4}, \quad d(u, u_n) < \frac{\delta}{4} \quad \forall\, n \ge N_2.$$

Hence, taking $\delta_0 := \delta/2$, we have

$$d(x,u) < \frac{\delta}{2} \quad \Longrightarrow \quad d(x_n, u_n) < \delta \quad \forall\, n \ge N_2$$

so that by (1),

$$\rho(f(x_n), f(u_n)) < \varepsilon \quad \forall\, n \ge N_2.$$

Therefore, by (2) and (3),

$$d(x,u) < \delta/2 \quad \Longrightarrow \quad \rho(f(x), f(u)) < 3\varepsilon \quad \forall\, n \ge N := \max\{N_1, N_2\}.$$

Thus, we have proved that f is continuous.

(iii) (Uniqueness of extension): We show that the continuous function f in (ii) is the unique continuous extension of f_0. Suppose $g : X \to Y$ is any continuous extension of f_0. Then for every $x \in X$, if (x_n) in X_0 is such that $d(x_n, x) \to 0$, then we have

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f_0(x_n) = f(x).$$

Thus, $g = f$. This proves the uniqueness of extension. $\square$

4.5 Homeomorphism, open and closed maps

Definition 4.33 Let X and Y be topological spaces, and $f : X \to Y$. Then f is said to be a **homeomorphism** if f is bijective, continuous, and its inverse $f^{-1} : Y \to X$ is continuous,. $\Diamond$

- Let X and Y be topological spaces, and $f : X \to Y$ be a bijective function. Then $f : X \to Y$ is a homeomorphism iff $f^{-1} : Y \to X$ is a homeomorphism

Definition 4.34 Let X and Y be topological spaces. If there is a homeomorphism from X onto Y, then X is said to be **homeomorphic** with Y, and in that case we write $X \sim Y$. $\Diamond$

We observe that
$$X \sim Y \iff Y \sim X.$$

- Let $\mathcal{X}$ be the family of all topological spaces. On $\mathcal{X}$, then the relation $\sim$ is an equivalence relation (Verify).

We shall represent the fact that X is not homeomorphic with Y by
$$X \not\sim Y.$$

Example 4.35 (A homeomorphism) Let $\mathbb{R}$ be with usual topology. For any $a, b \in \mathbb{R}$ with $a \neq 0$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = ax + b, \quad x \in \mathbb{R},$$
is a homeomorphism. It can be easily seen that f is a continuous function. Also, for $x, y \in \mathbb{R}$,
$$f(x) = y \iff x = \frac{y-b}{a},$$
so that f is bijective and f^{-1} is continuous. $\Diamond$

Example 4.36 (A continuous bijection which is not a homeomorphism) Let $X = \mathbb{R}$ be with discrete topology and $Y = \mathbb{R}$ with usual topology. Then the identity map $f : X \to Y$ defined by
$$f(x) = x, \quad x \in X,$$
is continuous and bijective, but not a homeomorphism: Clearly, f is bijective, and since X is with discrete topolgy, f is continuous as well. Note that, $g := f^{-1} : Y \to X$ is defined by $g(y) := f^{-1}(y) = y$ for every $y \in Y$, and the inverse image of the open set $\{1\}$ in X under the map g is not open in X. $\Diamond$

Example 4.37 (A homeomorphism between open intervals) For any a, b, c, d with $a < b$ and $c < d$, the open intervals (a, b) and (c, d) are homeomorphic with usual topologies on them: To see this consider the function $f : (a, b) \to (c, d)$ be defined by

$$f(x) = c + \frac{d-c}{b-a}(x-a).$$

Then we see that f is continuous, bijective, and its inverse is given by

$$f^{-1}(y) = a + \frac{b-a}{d-c}(y-c),$$

which is also continuous.

Using the same arguments, $[a, b]$ is homeomorphic with $[c, d]$, $[a, b)$ is homeomorphic with $[c, d)$ and $(a, b]$ is homeomorphic with $(c, d]$. $\Diamond$

> With respect to the usual topology on $\mathbb{R}$,
>
> - any two open intervals are homeomorphic.
> - any two closed and bounded intervals are homeomorphic.

For intervals with end points a and b, one may ask the following questions:

- Is $(a, b) \sim [a, b]$?
- Is $(a, b) \sim (a, b]$?
- Is $(a, b) \sim [a, b)$?
- Is $[a, b] \sim (a, b]$?
- Is $[a, b] \sim [a, b)$?
- Is $[a, b) \sim (a, b]$?

We shall see that answer to only the last question is in the affirmative.

Example 4.38 (Non-homeomorphic spaces) We have following.

(1) With topologies induced by the usual metric, the space $X = [a, b]$ is not homeomorphic with any of the spaces $Y_1 = (a, b)$, $Y_2 = (a, b]$, $Y_3 = [a, b)$: Note that X is compact, whereas Y_1, Y_2, Y_3 are not compact. Recall that, continuous

image of a compact metric space is compact. Hence, there is no continuous onto function from X to any of Y_1, Y_2, Y_3.

(2) With topologies induced by the usual metric, the space $X = (a, b)$ is not homeomorphic with any of the spaces $Y_1 = (a, b]$, $Y_2 = [a, b)$.

To see this, first consider X and Y_1. Assume for a moment that X and Y are homeomorphic. Then there is a continuous onto function $g : Y_1 \to X$. Let $x_0 = g(b)$. The

$$A = Y_1 \setminus \{b\} = (a, b), \quad g(A) = (a, x_0) \cup (x_0, b).$$

Note that A is a connected subset of Y_1 whereas $g(A)$ is disconnected, which is a contradiction to the fact that, continuous image of a connected set has to be connected. Thus, we have proved that $X = (a, b)$ is not homeomorphic with $Y_1 = (a, b]$.

Using similar arguments, it can be shown (Verify) that (a, b) is not homeomorphic with $[a, b)$. $\Diamond$

Example 4.39 (Homeomorphic spaces) With topologies induced by the usual metric on $\mathbb{R}$, the space $X = [a, b)$ is homeomorphic $Y = (a, b]$. To see this, let $\varphi : (a, b] \to [-b, -a)$ be defined by

$$\varphi(x) = -x.$$

This function is a homeomorphism from $(a, b]$ onto $[-b, -a)$, that is $[a, b) \sim [-b, -a)$. Hence, in view of Example 4.37, $[-b, -a) \sim [a, b)$. Hence, by the transitivity of the relation $\sim$, we have $[a, b) \sim (a, b]$. Since $(a, b] \sim (c, d]$, we also have $[a, b) \sim (c, d]$. $\Diamond$

$$\boxed{\begin{array}{lllll} (a,b) \sim (c,d), & [a,b) \sim [c,d), & (a,b] \sim (c,d], & (a,b] \sim [c,d), & [a,b] \sim [c,d] \\ (a,b) \not\sim [a,b], & (a,b) \not\sim [a,b), & (a,b) \not\sim (a,b], & (a,b] \not\sim [a,b], & [a,b) \not\sim [a,b]. \end{array}}$$

Remark 4.40 Bear in mind that, to show that two spaces are homeomorphic, it is enough to find one homeomorphism from X onto Y, whereas to show that two spaces are not homeomorphic, it is necessary to show that there is no homeomorphism from X onto Y. $\Diamond$

Example 4.41 (A homeomorphism) Let $(0,1)$ and $(1,\infty)$ be with usual topology, and $f:(0,1)\to(1,\infty)$ be defined by

$$f(x)=\frac{1}{x},\quad x\in(0,1).$$

We note that f is bijective with its inverse defined by

$$f^{-1}(y)=\frac{1}{y},\quad y\in(1,\infty).$$

Further, both f and f^{-1} are continuous (using a result from Real Analysis). Thus, f is a homeomorphism.

Now, let $c\in\mathbb{R}$. Let $f_c:(0,1)\to(c,\infty)$ be defined by

$$f_c(x)=c+\frac{1-x}{x}-1=c+\frac{1-x}{x},\quad x\in(0,1).$$

Then f_c is a homeomorphism from $(0,1)$ to (c,∞) (Verify). We know that (see Example 4.37), the space (a,b) is homeomorphic with $(0,1)$. Hence, it follows that (a,b) is homeomorphic with (c,∞).

Next, let $g_c:(c,\infty)\to(-\infty,c)$ be defined by

$$g(x)=-x,\quad x\in(c,\infty).$$

This is a homeomorphism from (c,∞) to $(-\infty,c)$. Hence, it follows that (a,b) is homeomorphic with $(-\infty,c)$.

Using similar arguments, it can be seen that $[a,b)$ is homeomorphic with $[c,\infty)$ and to $(-\infty,c]$ (Verify). ♢

Example 4.42 (A homeomorphism) Let $(-1,1)$ and $\mathbb{R}$ be with usual topology, and $f:(-1,1)\to\mathbb{R}$ be defined by

$$f(x)=\frac{x}{1-x^2},\quad x\in(-1,1).$$

This function is a homeomorphism: It can be easily seen that f is continuous (using a result from Real Analysis). Also, for $y\in\mathbb{R}$ and $x\in(-1,1)$,

$$f(x)=y\iff\frac{x}{1-x^2}=y\iff x=\frac{2y}{1+(1+4y^2)^{1/2}}.$$

This shows that f is bijective and the function $g:\mathbb{R}\to(-1,1)$ defined by

$$g(x)=\frac{2y}{1+(1+4y^2)^{1/2}},\quad y\in\mathbb{R}$$

is the inverse of f, and g is continuous (Verify).

We know that (see Example 4.37), the space (a,b) is homeomorphic with $(-1,1)$. Hence, it follows that (a,b) is homeomorphic with $\mathbb{R}$. ♢

Example 4.43 (A continuous bijection which is not a homeomorphism) Let $X = [0, 1)$ with usual topology and

$$Y = S^1 := \{(a, b) : a^2 + b^2 = 1\}$$

with subspace topology of $\mathbb{R}^2$ with usual topology. Let $f : X \to Y$ be defined by

$$f(t) = (\cos 2\pi t, \sin 2\pi t), \quad t \in [0, 1).$$

Then f is bijective and continuous, but its inverse is not continuous, and hence it is not a homeomorphism (See Appendix). $\Diamond$

Example 4.44 (Non-homeomorphic spaces) Let $X = [0, 1]$ and $Y = [0, \infty)$ be with usual topologies, so that X and Y are metric spaces. These spaces are not homeomorphic. In fact, there is no continuous onto function from X to Y, since X is compact and Y is not compact. $\Diamond$

Example 4.45 (Non-homeomorphic spaces) Let $X = \mathbb{R}$ with usual topology and $Y = \mathbb{R}$ with discrete topology. Then X and Y are not homeomorphic. In fact there is no continuous bijection function from X to Y. This is seen as follows: Suppose $f : X \to Y$ be a bijective function. Then inverse image of singleton sets under f are singleton sets. Now, singleton sets in Y are open sets, whereas singleton sets are not open in X. Hence, f cannot be continuous.

Also, if $g : Y \to X$ is a bijective function, then it is continuous, but its inverse is not continuous, by the same reason as in the last paragraph. $\Diamond$

Let $X \in \{(c, \infty), (-\infty, d), \mathbb{R}\}$ and $Y \in \{[c, \infty), (-\infty, d]\}$. Then

$(a, b) \sim X, \quad [a, b) \sim Y, \quad X \not\sim Y.$

Theorem 4.46 *Let X and Y be topological spaces, and $f : X \to Y$ be a homeomorphism. Then we have the following.*

(1) *Image of every open set in X is open in Y.*

(2) *Image of every closed set in X is closed in Y.*

Proof. We note that for every $A \subseteq X$,

$$(f^{-1})^{-1}(A) = f(A).$$

Indeed, since f is bijective, for $y \in Y$,

$$y \in (f^{-1})^{-1}(A) \iff f^{-1}(y) \in A \iff y \in f(A).$$

Hence, using the fact that f^{-1} is continuous, it follows that $f(A)$ is open if A is open, and $f(A)$ is closed if A is closed. $\square$

Functions having the properties (1) and (2) in Theorem 4.46 have special names.

Definition 4.47 Let X and Y be topological spaces, and $f : X \to Y$. Then a function $f : X \to Y$ is said to be

1. an **open map** if for every open set A in X, $f(A)$ is open in Y, and
2. a **closed map** if for every closed set A in X, $f(A)$ is closed in Y. $\Diamond$

- By Theorem 4.46, every homeomorphism is an open and closed map.
- If $f : X \to Y$ be a bijective continuous function between topological spaces, then f is open iff $f^{-1} : Y \to X$ is continuous.

> A bijective continuous function is a homeomorphism iff it is open.
>
> A bijective continuous function is a homeomorphism iff it is closed.

Example 4.48 The identity map on a topological space is open, closed, and a homeomorphism. $\Diamond$

Example 4.49 If X is a topological space and Y is a discrete space, then every function from X to Y is open and closed. $\Diamond$

Exercise 4.50 Write details of the proofs of the above two theorems.
(Hint: for every $A \subseteq X$, $(f^{-1})^{-1}(A) = f(A)$.) $\lhd$

Now, we give examples of following types of maps:

- An open map which is not a closed map;
- A closed map which is not an open map;
- A map which is neither open and nor closed;

Example 4.51 Let $X = (0, 1)$ and $Y = \mathbb{R}$ with usual metric topologies and let $f : X \to Y$ be the inclusion map. That is, $f(x) = x$, $x \in X$. This is an open map which is not a closed map:

Any open set in X is also open in Y; hence it is an open map. Note that, $A = (0, 1/2]$ is closed in X which is not closed in Y. Hence, f is not a closed map. $\Diamond$

Example 4.52 Let $X = [0, 1]$ and $Y = \mathbb{R}$ with usual metric topologies and let $f : X \to Y$ be the inclusion map. That is, $f(x) = x$, $x \in X$. This is an closed map which is not an open map:

Any closed set in X is also closed in Y; hence it is an closed map. Note that, $A = [0, 1/2)$ is open in X which is not open in Y. Hence, f is not an open map. $\Diamond$

Example 4.53 Let $X = [0, 1)$ and $Y = \mathbb{R}$ with usual metric topologies and let $f : X \to Y$ be the inclusion map. That is, $f(x) = x$, $x \in X$. This is an neither an open map nor a closed map: In this case $[0, 1)$ is open and closed in X, but it is neither open nor closed in Y. Hence, f is neither an open map nor a closed map. $\Diamond$

Example 4.54 Let $X = (0, 1)$ and $Y = (1, \infty)$, both with usual topology, and let $f : X \to Y$ be defined by

$$f(x) = \frac{1}{x}, \quad x \in X.$$

Then f is a homeomorphism,. $\Diamond$

Theorem 4.55 *Let X be a topological space and X_0 be a subset of X with subspace topology $\mathcal{T}_0$. Then the inclusion map $f : X_0 \to X$, defined by $f(u) = u$, $u \in X_0$, is*

(i) *open iff X_0 is open in X;*

(ii) *closed iff X_0 is closed in X.*

Proof. The proof follows from Theorem 2.40 and Theorem 2.46. $\square$

Exercise 4.56 Let $\mathbb{R}$ and $\mathbb{R}^2$ be with usual topologies. Show that the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) = (x, 0), \quad x \in \mathbb{R},$$

is a closed map, but not an open map. $\triangleleft$

Exercise 4.57 Verify the following:

1. The sets $A = \{(x, y) \in \mathbb{R}^2,\, x^2 + y^2 < 1\}$ and $B = \{(x, y) : |x| + |y| < 1\}$ are homeomorphic with respect to the subspace topologies induced by the usual topology on $\mathbb{R}^2$.

2. The sets $A = \{(x, y) \in \mathbb{R}^2,\, 1 < x^2 + y^2 < 2\}$ and $B = \{(x, y) : 1 < |x| + |y| < 2\}$ are homeomorphic with respect to the subspace topologies induced by the usual topology on $\mathbb{R}^2$.

3. The sets $A = \{(x, y) \in \mathbb{R}^2,\, x^2 + y^2 < 2\}$ and $B = \{(x, y) : 1 < |x| + |y| < 2\}$ are not homeomorphic with respect to the subspace topologies induced by the usual topology on $\mathbb{R}^2$.

4. The sets $A = \{(x, y) \in \mathbb{R}^2,\, x^2 + y^2 = 1\}$ and $B = \{x \in \mathbb{R} : 0 \le x \le 1\}$ are not homeomorphic with respect to the subspace topologies induced by the usual topology on the spaces $\mathbb{R}^2$ and $\mathbb{R}$, respectively. $\Diamond$

The following is an example of a bijective continuous fun ction, which is not a homeomorphism.

Example 4.58 Consider the Example 4.43. Since $t \mapsto \cos 2\pi t$ and $t \mapsto \sin 2\pi t$ are continuous on $[0, 1)$, the map $f : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ defined from $[0, 1)$ to $\mathbb{R}^2$ also continuous. Now, for $s, t \in [0, 1)$, we have

$$\begin{aligned}
f(t) = f(s) \quad &\Longleftrightarrow \quad (\cos 2\pi t, \sin 2\pi t) = (\cos 2\pi s, \sin 2\pi s) \\
&\Longleftrightarrow \quad (\cos 2\pi t - \cos 2\pi s, \sin 2\pi t - \sin 2\pi s) = (0, 0) \\
&\Longrightarrow \quad \cos 2\pi(s - t) = \cos 2\pi s \cos 2\pi t + \sin 2\pi s \sin 2\pi t \\
&\qquad\quad = \cos^2 2\pi t + \sin^2 2\pi t = 1 \\
&\Longrightarrow \quad 2\pi(s - t) = 0 \quad \Longrightarrow \quad s = t.
\end{aligned}$$

Thus, we have proved that $f : [0, 1) \to S^1$ is one-one. Now, we show that f is onto. For this, let $(a, b) \in S^1$ with $a \neq 0$,

$$(\cos 2\pi t, \sin 2\pi t) = (a, b) \quad \Longrightarrow \quad \tan 2\pi t = \frac{b}{a} \quad \Longrightarrow \quad t = \frac{1}{2\pi} \tan^{-1}(b/a).$$

If $(a, b) \in S^1$ with $a = 0$, we have $b = 1$ so that $f(0) = (0, 1)$. Thus, we have shown that $f : [0, 1) \to S^1$ is bijective and continuous. However, its inverse is not continuous. To see this, consider the open set $[0, 1/4)$ in $[0, 1)$. Its inverse image under f^{-1} is

$$\{(\cos 2\pi t, \sin 2\pi t) : 0 \le t < 1/4\}$$

which is not an open set in S^1. $\Diamond$

4.6 Topologies determined by functions

Let X and Y be sets and $f : X \to Y$. Then we have the following:

- $\varnothing = f^{-1}(\varnothing)$ and $X = f^{-1}(Y)$;
- For subsets A_1, A_2 of Y, $f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2)$;
- For a family $\{A_\alpha : \alpha \in \Lambda\}$ of subsets of Y, $\displaystyle\bigcup_{\alpha\in\Lambda} f^{-1}(A_\alpha) = f^{-1}\Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)$.

The above observations show that if X is a set, $(Y, \mathcal{T}_Y)$ is a topological space, then any function $f : X \to Y$ determines a topology on X, given by

$$\mathcal{T}_{f,X} := \{f^{-1}(G) : G \text{ open in } X\}.$$

Clearly, f is continuous with respect to $(\mathcal{T}_{f,X}, \mathcal{T}_Y)$. Further, if there is another topology $\mathcal{T}$ on X such that f is continuous with respect to $(\mathcal{T}, \mathcal{T}_Y)$, then $\mathcal{T}_{f,X}$ is weaker than $\mathcal{T}$. Thus, we have proved the following theorem.

Theorem 4.59 *Let X be a set, Y be a topological space, and $f : X \to Y$. Then*

$$\mathcal{T}_{f,X} = \{f^{-1}(G) : G \textit{ open in } Y\}$$

is a topology on X, and with this topology on X, $f : X \to Y$ is continuous. Further, $\mathcal{T}_{f,X}$ is the weakest topology on X such that f is continuous.

Definition 4.60 The topology $\mathcal{T}_{f,X}$ in Theorem 4.59 is called the **weak topology on X determined by f**. $\Diamond$

Now, suppose $(X, \mathcal{T}_X)$ is a topological space and Y is set. Can we define a topology on Y using a function $f : X \to Y$ in some manner?

Here is an answer.

Theorem 4.61 *Let X be be a topological space, Y be a set, and $f : X \to Y$.*

$$\mathcal{T}_{f,Y} = \{A \subseteq Y : f^{-1}(A) \textit{ open in } X\}$$

is a topology on Y, and with this topology on Y, $f : X \to Y$ is continuous. Further, $\mathcal{T}_{f,Y}$ is the strongest topology on Y such that f is continuous.

Proof. We observe that $\varnothing \in \mathcal{T}_{f,Y}$ since $\varnothing = f^{-1}(\varnothing)$ is open open in X, and $Y \in \mathcal{T}_{f,Y}$ since $X = f^{-1}(Y)$ is open in X. Also, if $A, B \in \mathcal{T}_{f,Y}$, then $f^{-1}(A)$ and $f^{-1}(B)$ are open in X so that

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

open in X, and hence $A \cap B \in \mathcal{T}_{f,Y}$.

Next, let $\{A_\alpha : \alpha \in \Lambda\} \subseteq \mathcal{T}_{f,Y}$ for some index set Λ. Then

$$f^{-1}(A_\alpha) \text{ open in } X \quad \forall\, \alpha \in \Lambda$$

so that

$$f^{-1}\Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big) = \bigcup_{\alpha\in\Lambda} f^{-1}(A_\alpha) \text{ open in } X.$$

Hence, $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{T}_{f,Y}$. Thus, we have proved that $\mathcal{T}_{f,Y}$ is a topology on Y.

Now, suppose that $\mathcal{T}$ is a topology on Y such that f is continuous with respect to $(X, \mathcal{T}_{f,Y})$. Then, for every $G \in \mathcal{T}$, $f^{-1}(G)$ is open in X. Hence, $G \in \mathcal{T}_{f,Y}$. This shows that $\mathcal{T}_{f,Y}$ is stronger than $\mathcal{T}$. Thus, we have proved that $\mathcal{T}_{f,Y}$ is the strongest topology on Y such that f is continuous. $\square$

Definition 4.62 If X is a topological space and Y is a set, then the topology $\mathcal{T}_{f,Y}$ obtained as in Theorem 4.61 is called the **strong topology on** Y **determined by** f. $\Diamond$

Example 4.63 Let X be a nonempty set, Y be a topological space and $y_0 \in Y$. Let $f : X \to Y$ be the constant function,

$$f(x) = y_0, \quad x \in X.$$

We note that, for any open set $G \subseteq Y$,

$$f^{-1}(G) = \begin{cases} X, & y_0 \in G, \\ \varnothing, & y_0 \notin G. \end{cases}$$

Thus, the weak topology on X determined by f is the indiscrete topology. $\Diamond$

Example 4.64 Let X be a topological space, Y be a set and $y_0 \in Y$. Let $f : X \to Y$ be the constant function,

$$f(x) = y_0, \quad x \in X.$$

Now, if A is any subset of Y, then either $y_0 \in A$ or $y_0 \notin A$. If $y_0 \in A$, then $f^{-1}(A) = X$ and if $y_0 \notin A$, then $f^{-1}(A) = \varnothing$. Thus, we have proved that every subset of A is open w.r.t. the topology on Y determined by f. In other words, the topology on Y determined by f is the discrete topology. $\Diamond$

Recall that if X and Y are topological spaces, and $f : X \to Y$ is a continuous function, then for any sequence (x_n) in X,

$$x_n \to x \quad \Longrightarrow \quad f(x_n) \to f(x).$$

The next theorem shows that, if the topology on X is determined by f, then the reverse implication is also true.

Theorem 4.65 *Let X be a set, $(Y, \mathcal{T}_Y)$ be a topological space. Let $\mathcal{T}_{f,X}$ be the weak topology on X determined by a function $f : X \to Y$. Then, for any sequence (x_n) in X, $x_n \to x$ with respect to $\mathcal{T}_{f,X}$ iff $f(x_n) \to f(x)$ with respect to $\mathcal{T}_Y$.*

Proof. Since f is continuous with respect to $(\mathcal{T}_{f,X}, \mathcal{T}_Y)$, we have the implication

$$x_n \to x \text{ in } (X, \mathcal{T}_{f,X}) \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } (Y, \mathcal{T}_Y).$$

To see the reverse implication, suppose (x_n) is a sequence in X such that $f(x_n) \to f(x)$ for some $x \in X$. We have to show that $x_n \to x$ with respect to $\mathcal{T}_{f,X}$. For this, let G be an open neighbourhood of x. Then, by the definition of $\mathcal{T}_{f,X}$, there exists an open set V in Y such that $G = f^{-1}(V)$. Since $x \in G = f^{-1}(V)$, we have $f(x) \in V$. Now, $f(x_n) \to f(x)$ implies that there exists $N \in \mathbb{N}$ such that

$$f(x_n) \in V \quad \forall\, n \geq N.$$

Hence, $x_n \in f^{-1}(V) = G$ for all $n \geq N$, showing that $x_n \to x$. $\square$

Instead of having topologies determined by a single function, we can have topologies determined by families of functions. Thus, we have the following two theorems, which are more general than Theorem 4.59 and Theorem 4.61, respectively.

Theorem 4.66 *Let X be a set, $\{(Y_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, and $f_\alpha : X \to Y_\alpha$ for $\alpha \in \Lambda$. Then the topology generated by*

$$\mathcal{S} := \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \mathcal{T}_\alpha,\, \alpha \in \Lambda\}$$

is the weakest topology on X such that each f_α is continuous.

Proof. Let $\mathcal{T}_X$ be the topology generated by $\mathcal{S}$. Clearly, each f_α is continuous with respect to $(\mathcal{T}_X, \mathcal{T}_\alpha)$. Now, let $\mathcal{T}$ be any other topology on X such that each f_α is continuous with respect to $(\mathcal{T}, \mathcal{T}_\alpha)$. Then, $f_\alpha^{-1}(G_\alpha) \in \mathcal{T}$ for each $G_\alpha \in \mathcal{T}_\alpha$. Hence, $\mathcal{S} \subseteq \mathcal{T}$, so that $\mathcal{T}_X \subseteq \mathcal{T}$. $\square$

Theorem 4.67 *Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, Y be a set and $f_\alpha : X_\alpha \to Y$ for $\alpha \in \Lambda$. Then*

$$\mathcal{T}_Y := \{A \subseteq Y : f_\alpha^{-1}(A) \in \mathcal{T}_\alpha \textit{ forall } \alpha \in X\}$$

is the strongest topology on Y such that each $f_\alpha : X_\alpha \to Y$ is continuous.

Proof. Clearly, $f_\alpha^{-1}(\varnothing) = \varnothing$ and $f_\alpha^{-1}(Y) = X$ so that $\varnothing, Y \in \mathcal{T}_Y$. Also, for $A, B \in \mathcal{T}_Y$,

$$f_\alpha^{-1}(A \cap B) = f_\alpha^{-1}(A) \cap f_\alpha^{-1}(A) \in \mathcal{T}_\alpha,$$

for each $\alpha \in \Lambda$, so that $A \cap B \in \mathcal{T}_Y$. Further, if $\mathcal{G}$ is a subfamily of $\mathcal{T}_Y$, then

$$f_\alpha^{-1}\Big(\bigcup_{A \in \mathcal{G}} A\Big) = \bigcup_{A \in \mathcal{G}} f_\alpha^{-1}(A) \in \mathcal{T}_\alpha$$

for each $\alpha \in \Lambda$, so that $\bigcup_{A \in \mathcal{G}} A \in \mathcal{T}_Y$. Thus, $\mathcal{T}_Y$ is a topology on Y.

Clearly, for every $A \in \mathcal{T}_Y$, $f_\alpha^{-1}(A) \in \mathcal{T}_\alpha$ for every $\alpha \in \Lambda$ so that each f_α is continuous with respect to $(\mathcal{T}_\alpha, \mathcal{T}_Y)$. Now, let $\mathcal{T}$ be any other topology on Y such that each f_α is continuous with respect to $(\mathcal{T}_\alpha, \mathcal{T})$. Then, by the definition of continuity, for every $A \in \mathcal{T}$, $f_\alpha^{-1}(A) \in \mathcal{T}_\alpha$ for every $\alpha \in \Lambda$, so that $A \in \mathcal{T}_Y$. Thus, we have proved that $\mathcal{T} \subseteq \mathcal{T}_Y$, showing that $\mathcal{T}_Y$ is the strongest topology on Y such that each $f_\alpha : X_\alpha \to Y$ is continuous. $\square$

Definition 4.68 The topology on X in Theorem 4.66 is called the **weak topology on X determined by** $\{f_\alpha : X \to Y_\alpha | \alpha \in \Lambda\}$. $\Diamond$

Definition 4.69 The topology on Y in Theorem 4.67 is called the **strong topology on Y determined by** $\{f_\alpha : X_\alpha \to Y | \alpha \in \Lambda\}$. $\Diamond$

Chapter 5

Connected and Path Connected Spaces

In real analysis, we learned the notion of connectedness within metric spaces. We now extend this concept to general topological spaces, demonstrating that the core properties of connectedness depend entirely on open sets rather than an explicit distance metric.

5.1 Connected space

Definition 5.1 A topological space X is said to be a **connected space** if it is not a union of two disjoint nonempty open sets. A topological space which is not connected is called a **disconnected space**. $\diamondsuit$

Definition 5.2 By a **separation** of a topological space X, we mean, a pair (A, B) of disjoint nonempty open sets in X such that $X = A \cup B$. $\diamondsuit$

- A topological space X is connected iff there is no separation of X.
- A topological space X is disconnected iff there is a separation of X.

Definition 5.3 A subset of a topological space is said to be **connected** if it is a connected space as a subspace. $\diamondsuit$

Example 5.4 The following can be verified easily (Exercise).

1. A subset of a discrete space is connected iff it is a singleton set.
2. If X is an uncountable set with co-countable topology, then X is connected.
3. If X is an infinite set with co-finite topology, then X is connected. $\diamondsuit$

Theorem 5.5 *A nonempty topological space is connected iff the whole space is its only nonempty subset which is both open and closed.*

Proof. Let X nonempty topological space. Suppose X is connected, and E is a nonempty subset which is both open and closed. Then E^c is also both open and closed. Since X is connected and $X = E \cup E^c$, we have $E^c = \varnothing$, for otherwise, (E, E^c) would be a separation of X. Hence, $E = X$.

Conversely, suppose that X is the only nonempty subset which is both open and closed. We have to show that X is connected. Assume on the contrary that X is not connected, and (A, B) is a separation of X. Then A and B are different subsets which are both open and closed, which is a contradiction to the hypothesis. □

Theorem 5.6 *Let X be a topological space and $E \subseteq X_0 \subseteq X$. Then E is a connected subset of X_0 iff E is a connected subset of X.*

Proof. It is enough to show that E disconnected in X_0 iff E is disconnected in X.

Suppose E disconnected in X_0. Hence, there exists open sets V_1 and V_2 in X_0 such that

$$E = (E \cap V_1) \cup (E \cap V_2),$$

where $E \cap V_1$ and $E \cap V_2$ are nonempty disjoint sets. Since V_1 and V_2 are open in X_0, there exists open sets G_1 and G_2 in X such that $V_1 = X_0 \cap G_1$ and $V_2 = X_0 \cap G_2$. Hence,

$$E \cap V_1 = E \cap X_0 \cap G_1 = E \cap G_1, \quad E \cap V_2 = E \cap X_0 \cap G_2 = E \cap G_2.$$

Thus, $E \cap G_1$ and $E \cap G_2$ are disjoint nonempty subsets of E such that

$$E = (E \cap G_1) \cup (E \cap G_2),$$

where G_1 and G_2 are open in X. This shows that E is disconnected in X.

Conversely, suppose that E disconnected in X. Then there exists open sets G_1 and G_2 in X such that

$$E = (E \cap G_1) \cup (E \cap G_2),$$

where $E \cap G_1$ and $E \cap G_2$ are nonempty disjoint sets. Note that

$$E \cap G_1 = E \cap X_0 \cap G_1 = E \cap V_1, \quad E \cap G_2 = E \cap X_0 \cap G_2 = E \cap V_2,$$

where $V_1 = X_0 \cap G_1$ and $V_2 = X_0 \cap G_2$. Hence,

$$E = (E \cap V_1) \cup (E \cap V_2),$$

where $E \cap V_1$ and $E \cap V_2$ are nonempty disjoint sets with V_1 and V_2 are open in X_0. This shows that E is disconnected in X_0. □

Remark 5.7 In view of the above theorem, while talking about connected sets in a topological space, we do not specify explicitly whether it is connected in the whole space or in a subspace. ♢

Theorem 5.8 *Continuous image of a connected set is connected.*

Proof. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. Let E be a connected subset of X. To show that $f(E)$ is connected subset of Y.

Assume for a moment that $f(E)$ is not connected subset of Y. Then there exists disjoint nonempty open sets V_1 and V_2 in $f(E)$ such that $f(E) = V_1 \cup V_2$. Let G_1 and G_2 be open sets in Y such that $V_1 = G_1 \cap f(E)$ and $V_2 = G_2 \cap f(E)$. Then we have

$$f(E) \subseteq G_1 \cup G_2$$

so that

$$E \subseteq f^{-1}(G_1) \cup f^{-1}(G_2),$$

and hence

$$E = [E \cap f^{-1}(G_1)] \cup [E \cap f^{-1}(G_2)].$$

Note that, since V_1 and V_2 are disjoint nonempty sets implies that $E \cap f^{-1}(G_1)$ and $E \cap f^{-1}(G_2)$ are are disjoint nonempty sets in E, contradicting the fact that E is connected. □

Example 5.9 Let $X = \mathbb{R}$ with usual topology and Y be a discrete space. Then the only continuous function from X to Y is the constant function: Suppose $f : X \to Y$ is a continuous function. Then $f(X)$ is connected, as X is connected. Since Y is a discrete space, $f(X)$ has to be a singleton set. Thus f is a constant function. ♢

Theorem 5.10 *Let A and B be nonempty connected subsets of a topological space X such that $A \cap B \neq \varnothing$. Then $A \cup B$ is connected in X.*

This is a particular case of the following more general result.

Theorem 5.11 *If $\{A_\alpha : \alpha \in \Lambda\}$ is a family of connected subsets of a topological space X such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \varnothing$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is connected.*

Proof. Let $x_0 \in \bigcap_{\alpha\in\Lambda} A_\alpha$. Suppose $X_0 := \bigcup_{\alpha\in\Lambda} A_\alpha$ is not connected. Then there exists disjoint nonempty open sets V_1 and V_2 in X_0 such that $X_0 = V_1 \cup V_2$. Let G_1 and G_2 be open sets in X such that $V_1 = G_1 \cap X_0$ and $V_2 = G_2 \cap X_0$. Since $V_1 \cap V_2 = \varnothing$, either $x_0 \in V_1$ or $x_0 \in V_2$, but not in both. Assume that $x_0 \in V_1$.

Now, let $\alpha \in \Lambda$. Then

$$A_\alpha = A_\alpha \cap (V_1 \cup V_2) = (A_\alpha \cap V_1) \cup (A_\alpha \cap V_2), \qquad (*)$$

where $A_\alpha \cap V_1$ and $A_\alpha \cap V_2$ are disjoint open sets in A_α. Since A_α is connected and $x_0 \in A_\alpha \cap V_1$, we obtain $A_\alpha \cap V_2 = \varnothing$. Hence, $A_\alpha = A_\alpha \cap V_1 \subseteq V_1$. This is true for every $\alpha \in \Lambda$. Hence

$$X_0 = \bigcup_{\alpha \in \Lambda} A_\alpha \subseteq V_1 \subseteq X_0.$$

This is a contradiction to the fact that $V_2 \neq \varnothing$. $\square$

As an immediate corollary to Theorem 5.10, we have the following result.

Theorem 5.12 *If $E_0, E_1, \ldots, E_n$ are connected subsets of a topological space X such that $E_{i-1} \cap E_i \neq \varnothing$ for $i = 1, \ldots, n$, then $\bigcup_{i=0}^n E_i$ is connected.*

A natural question one may ask is the following:

> Suppose (E_n) is a sequence of connected subsets of a topological space X such that $E_n \cap E_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Is $\bigcup_{n=1}^\infty E_n$ connected?

We shall answer this question affirmatively by making use of the following proposition.

Proposition 5.13 *Suppose X is a disconnected topological space and (A, B) is a separation of X. If E is a connected subset of X, then either $E \subseteq A$ or $E \subseteq B$.*

Proof. Given that $X = A \cup B$, where A and B are disjoint nonempty open subsets of X such that $A = A \cup B$. Then we have

$$E = (E \cap A) \cup (E \cap B). \qquad (*)$$

Note that $E \cap A$ and $E \cap B$ are open in E, and they are disjoint as well. Hence, by the connectedness of E,

$$E \cap A = \varnothing \text{ or } E \cap B = \varnothing.$$

If $E \cap A = \varnothing$, then from $(*)$, we see that $E = E \cap B$, so that $E \subseteq B$. Similarly, $E \cap B = \varnothing$ implies $E \subseteq A$. $\square$

As a corollary to the above, we have the following.

Theorem 5.14 *Let X be a topological space. Then X is connected iff every pair of points in X is contained in a connected subset of X.*

Proof. Clearly, if X is connected, then every pair of points in X is contained in the connected set X.

Conversely, suppose that every pair of points in X is contained in an connected subset of X. We have to show that X is connected. Assume on the contrary that X is not connected. Let (A, B) be a separation of X. Let $x \in A$ and $y \in B$. By hypothesis, there exists a connected set E containing both x and y. By Proposition 5.13, either $E \subseteq A$ or $E \subseteq B$. In particular, $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq B$. It is a contradiction to the fact that A and B are disjoint and the way x, y are chosen. $\square$

Now, we state and prove the promised result.

Theorem 5.15 (Chain theorem) *Suppose (E_n) is a sequence of connected subsets of a topological space X such that $E_n \cap E_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} E_n$ connected.*

Proof. By Theorem 5.14, it is enough to show that every pair of points in X is contained in a connected subset of $E := \bigcup_{n=1}^{\infty} E_n$. So, let $x, y \in E$. Then there exists $n_x, n_y \in \mathbb{N}$ such that $x \in E_{n_x}$ and $y \in E_{n_y}$. Let $N = \max\{n_x, n_y\}$. Then

$$\{x, y\} \subseteq \bigcup_{n=1}^{N} E_n.$$

By Theorem 5.12, $\bigcup_{n=1}^{N} E_n$ is connected. Thus, we have shown that there is a connected set containing $\{x, y\}$, completing the proof. $\square$

Theorem 5.16 (Intermediate value theorem) *Let X be a connected topological space and $f : X \to \mathbb{R}$ be a continuous function (with usual topology on $\mathbb{R}$). Suppose $x_1, x_2 \in X$ be such that $f(x_1) < f(x_2)$. Then, for every $\gamma \in \mathbb{R}$ such that*

$$f(x_1) < \gamma < f(x_2),$$

there exists $x \in X$ such that

$$f(x) = \gamma.$$

Proof. Let $\gamma \in \mathbb{R}$ be such that $f(x_1) < \gamma < f(x_2)$. Suppose there is no $x \in X$ such that $f(x) = \gamma$. Then we see that

$$X = \{x \in X : f(x) < \gamma\} \cup \{x \in X : f(x) > \gamma\}.$$

Since f is continuous, the sets $A = \{x \in X : f(x) < \gamma\}$ and $B = \{x \in X : f(x) > \gamma\}$ are open sets. Clearly,

$$x_1 \in A, \quad x_2 \in B, \quad A \cap B = \varnothing.$$

Thus, we arrive at a contradiction to the fact that X is connected. $\square$

We recall from *real analysis* that a subset of $\mathbb{R}$ is connected iff it is an interval. Now, we prove an analogous result in $\mathbb{R}^n$. First, let us recall that a subset E of $\mathbb{R}^n$ is **convex** if the line segment joining any tow points in E is contained in E, that is, for every $x, y \in E$,

$$\{(1-t)x + ty : 0 \le t \le 1\} \subseteq E.$$

Theorem 5.17 *Every convex subset of $\mathbb{R}^n$ is connected with respect to the usual topology on $\mathbb{R}^n$.*

Proof. Let E be a convex subset of $\mathbb{R}^n$. By Theorem 5.14, it is enough to prove that every pair of points in E is contained in a connected subset of E. So, let $x, y \in E$. Let $f : [0,1] \to X$ be defined by

$$f(t) = (1-t)x + ty, \quad t \in [0,1].$$

Then f is continuous and $[0,1]$ is connected, its range, namely, the set

$$f([0,1]) = \{(1-t)x + ty : 0 \le t \le 1\}$$

is connected. Since E is convex, $\{(1-t)x+ty : 0 \le t \le 1\} \subseteq E$. Thus, $f([0,1])$ is a connected subset of E containing x, y. $\square$

Theorem 5.18 *Closure of a connected subset is connected.*

Proof. Let E be a connected subset of a topological space X. We have to show that $\bar{E}$ is connected. Assume, on the contrary that $\bar{E}$ is not connected. Let (A, B) be a separation of $\bar{E}$. Hence,

$$E \subseteq \bar{E} = A \cup B = (\bar{E} \cap G_1) \cup (\bar{E} \cap G_2)$$

for some open sets G_1 and G_2. Hence,

$$E = (E \cap G_1) \cup (E \cap G_2). \tag{$*$}$$

Note that

$$\bar{E} \cap G_1 \ne \varnothing \quad \Longrightarrow \quad E \cap G_1 \ne \varnothing, \quad \bar{E} \cap G_2 \ne \varnothing \quad \Longrightarrow \quad E \cap G_2 \ne \varnothing.$$

(If $x \in \bar{E} \cap G_1$, then the open set G_1 which contains a closure pint x of E must contain some point of E.) Thus, $(*)$ shows that E has a separation, contradicting the connectedness of E. $\square$

More generally, we have the following.

Theorem 5.19 *Suppose A is a connected subset of a topological space X and $B \subseteq X$ is such that*

$$A \subseteq B \subseteq \bar{A}.$$

Then B is connected.

Proof. Suppose B is not connected. Let (B_1, B_2) be a separation of B. Then there exist open sets G_1 and G_2 in X such that

$$B = (B \cap G_1) \cup (B \cap G_2),$$

with $B \cap G_1$ and $B \cap G_2$ are nonempty and disjoint. This implies

$$A = (A \cap G_1) \cup (A \cap G_2),$$

Note that

$$B \cap G_1 \neq \varnothing \quad \Longrightarrow \quad \bar{A} \cap G_1 \neq \varnothing \quad \Longrightarrow A \cap G_1 \neq \varnothing.$$

Similarly,

$$B \cap G_2 \neq \varnothing \quad \Longrightarrow \quad \bar{A} \cap G_2 \neq \varnothing \quad \Longrightarrow A \cap G_2 \neq \varnothing.$$

Hence, $(A \cap G_1, A \cap G_2)$ is s separation of A, contradicting the connectedness of A. $\square$

Theorem 5.20 *A topological space is disconnected iff there exists a continuous function from X onto a two-point set with discrete topology.*

Proof. Let X be a topological space. Suppose X is disconnected. Then there exists disjoint nonempty open subsets A and B such that $X = A \cup B$. Let $\{0, 1\}$ be with discrete topology and $f : X \to \{0, 1\}$ be defined by

$$f(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in B. \end{cases}$$

Then f is onto. Since A and B are also closed sets, by *pasting lemma*, f is continuous.

Conversely, suppose $f : X \to \{0, 1\}$ be a surjective continuous function, where $\{0, 1\}$ is endowed with discrete topology. Let

$$A = \{x \in X : f(x) = 0\} \quad \text{and} \quad B = \{x \in X : f(x) = 1\}.$$

Then A and B are nonempty open sets and $X = A \cup B,$. Hence, X is disconnected. $\square$

5.2 Totally disconnected space

Definition 5.21 A topological space is said to be **totally disconnected** if singleton sets are its only connected subsets. $\Diamond$

Example 5.22 Every discrete space is totally disconnected. Indeed, if X is a discrete space and $E \subseteq X$ contains more than one elements, then for every $x \in E$, $(\{x\}, E \setminus \{x\})$ is a separation of E. $\Diamond$

Example 5.23 The topological spaces $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected with respect to the induced usual topology. To see that $\mathbb{Q}$ is totally connected, let $E \subseteq \mathbb{Q}$ contains more than one elements. Let $x, y \in E$ be such that $x < y$, and let $r \in \mathbb{R} \setminus \mathbb{Q}$ be such that $x < r < y$. Let $G_1 = (-\infty, r)$ and $G_2 = (r, \infty)$. Then we have

$$E = (E \cap G_1) \cup (E \cap G_2).$$

Note that $E \cap G_1$ and $E \cap G_2$ are disjoint open subsets E containing x and y, respectively. Hence, E is disconnected.

Analogously, it can be shown that every subset of $\mathbb{R} \setminus \mathbb{Q}$ containing more than one elements is disconnected. $\Diamond$

5.3 Path connectedness

For proving that every convex subset of $\mathbb{R}^n$ is connected, what we have actually done was that every pair of points in $\mathbb{R}^n$ can be joined by a straight line segment and used theorem Theorems 5.8 and 5.14. Similar idea can be used prove connectedness of more general sets in topological spaces, as the following theorem shows.

Theorem 5.24 *Let X be a topological space such that for every $x, y \in X$, there exists a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. Then X is connected.*

Proof. Under the given assumption, we have to show that X is connected. By Theorem 5.14, it is enough to show that every pair of points in X is contained in a connected subset of X. So, let $x, y \in X$. By the assumption on X, there exists a continuous function $f : [a, b] \to X$ for some $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = x$ and $f(b) = y$. Since $[a, b]$ is connected and f is continuous, the set $f([a,b])$ is connected in X with $x = f(a) \in f([a,b])$ and $y = f(b) \in f([a,b])$, and the proof is complete. $\square$

Definition 5.25 Let X be a topological space and $x, y \in X$. By a **path** from x to y, we mean, a continuous function $f : [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$, and in that case, we say that x and y are joined by a path. $\Diamond$

We observe that:

- If $f : [a,b] \to X$ is a path from x to y, then $g : [a,b] \to X$ defined by

$$g(t) = f(a + b - t), \quad t \in [a,b],$$

  is a path from y to x.

Definition 5.26 A topological space X is said to be a **path connected space** if any pair of points in X can be joined by a path in X. $\Diamond$

In view of the above two definitions, Theorem 5.24 can be re-stated as follows.

Theorem 5.27 *Every path connected space is connected.*

Exercise 5.28 Every convex subset of $\mathbb{R}^n$ is path connected with respect to the usual topology. $\triangleleft$

Let X be a topological space and $x, y \in X$. If $f : [a,b] \to X$ is a continuous function such that $f(a) = x$ and $f(b) = y$, then for any closed interval $[c,d]$, there is a continuous function $g : [c,d] \to X$ such that $g(c) = x$ and $g(d) = y$. This is seen as follows:

Consider the function $h : [c,d] \to [a,b]$ defined by

$$h(t) = a + \frac{b-a}{d-c}(t-c), \quad t \in [c,d].$$

Then, h is a continuous function such that $h(c) = a$ and $h(d) = b$. Now, consider the function $g := f \circ h$, that is, $g : [c,d] \to X$ defined by

$$g(t) = f(h(t)), \quad t \in [c,d].$$

Then g is continuous and it satisfies

$$g(c) = f(h(c)) = f(a) = x \quad \text{and} \quad g(d) = f(h(d)) = f(b) = y.$$

Thus, we have shown the following.

Proposition 5.29 *Let x, y be in a topological space X. If $f : [a, b] \to X$ is a path from x to y and if $h : [c, d] \to [a, b]$ defined by*

$$h(t) = a + \frac{b-a}{d-c}(t-c), \quad t \in [c, d],$$

then $g : [c, d] \to X$ defined by

$$g(t) = f(h(t)), \quad t \in [c, d],$$

is also a path from x to y.

Proposition 5.30 *Let X be a topological space and $x, y, z \in X$. If there is a path from x to y and a path from y to z, then there is a path from x to z.*

Proof. Let $f : [a, b] \to X$ and $g : [a, b] \to X$ be paths from x to y and y to z, respectively. Then, by Proposition 5.29, there are paths

$$f_1 : [0, 1] \to X \quad \text{and} \quad f_2 : [1, 2] \to X$$

from x to y and y to z, respectively. Next, define $g : [0, 2] \to X$ by

$$g(t) = \begin{cases} f_1(t) & \text{if } t \in [0, 1], \\ f_2(t) & \text{if } t \in [1, 2]. \end{cases}$$

Since f_1 and f_2 are continuous functions defined on closed subsets of $\mathbb{R}$ (with respect to the usual topology) and $f_1(1) = f_2(1) = y$, by pasting lemma (Theorem 4.29), g is continuous. Also, we have $g(0) = x$ and $g(2) = z$. Thus, g is a path from x to z. $\square$

- Converse of Theorem 5.27 is not true, as the following example shows.

Example 5.31 (See Munkres: Example 6, Sec. 2.4, Chapter 3.)
Let $X = \mathbb{R}^2$ with usual topology, and let

$$E = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

Note that E is connected, as it is the continuous image of an open interval. Hence, we know that its closure $\bar{E}$ is connected. We show that $\bar{E}$ is not path connected.

Note that $\bar{E} = E \cup F$, where $F = \{(x, y) \in \mathbb{R}^2 : x = 0, -1 \leq y \leq 1\}$. We show that there is no path from the origin to any point E. For this, assume on the contrary that there is a continuous function $f : [a, b] \to \bar{E}$ such $f(a)$ is the origin and $f(b)$ is a particular point in E. Consider the the set

$$J = \{t \in [a, b] : f(t) \in F\}.$$

Since J is closed and bounded, it has a maximum in J, say $c = \max J$. Now, consider the restriction map $f : [c, b] \to \bar{E}$. Note that $f(c) \in F$ and $f(b) \in E$. Let us assume, without loss of generality, that $[c, b] = [0, 1]$, and write

$$f(t) = (x(t), y(t)), \quad 0 \le t \le 1.$$

Then, we have $x(0) = 0$ and for $0 < t \le 1$, $y(t) = \sin(1/x(t))$. Our idea is to find a sequence (t_n) in the open interval $(0, 1)$ such that $t_n \to 0$, but $(x(t_n))$ does not converge, which would contradict the continuity of f at 0. For this, for $n \in \mathbb{N}$, let s_n such that $0 < s_n < x(1/n)$ and $\sin(1/s_n) = (-1)^n$. Since $x(\cdot)$ is continuous, there exists t_n such that

$$0 < t_n < \frac{1}{n} \quad \text{and} \quad x(t_n) = s_n.$$

Thus, we have $t_n \to 0$ but $(y(t_n))$ does not converge. $\Diamond$

Remark 5.32 The set $\bar{E}$ in Example 5.31 is called the *topologist's sine curve.* $\Diamond$

Theorem 5.33 *Let A and B be path connected subsets of a topological space such that $A \cap B \ne \varnothing$. Then $A \cup B$ is path connected in X.*

Proof. Let $x, y \in A \cup B$. Suppose both x and y are either in A or in B. Since A are path connected, it follows that there is a path joining x and y.

Next, suppose that $x \in A$ and $y \in B$. Let $z \in A \cap B$. Since $x, z \in A$, there is a path from x to z, and since $z, y \in B$, there is a pat from z to y. Therefore, there is a path from x to y. $\Box$

Corollary 5.34 *Let $\{E_\alpha : \alpha \in \Lambda\}$ be a family of path connected subsets of a topological space X such that $\bigcap_{\alpha\in\Lambda} E_\alpha \ne \varnothing$. Then $E := \bigcup_{\alpha\in\Lambda} E_\alpha$ is path connected.*

Proof. Let $x_0 \in \bigcap_{\alpha\in\Lambda} E_\alpha$, and let $x, y \in E := \bigcup_{\alpha\in\Lambda} E_\alpha$. Let $\beta, \gamma \in \Lambda$ be such that $x \in A_\beta$ and $y \in A_\gamma$. Since $x_0, x \in A\beta$ and $x_0, y \in A_\gamma$, and since A_β and A_γ are path connected, there is a path in A_β from x to x_0 and there is a path in A_γ from x_0 to y. In particular, there is a path in E from x to x_0 and there is a path in E from x_0 to y. Hence, there is a path in E from x to y. $\Box$

The following two theorems are also immediate consequences of Theorem 5.33, and the details of their proofs are left as exercises.

Theorem 5.35 *Let $E_0, E_1, \ldots, E_n$ be path connected subsets of a topological space X such that $E_{i-1} \cap E_i \ne \varnothing$ for $i \in \{1, \ldots, n\}$. Then $\bigcup_{i=0}^{n} E_i$ is path connected.*

Theorem 5.36 *Let (E_n) be a sequence of path connected subsets of a topological space X such that $E_n \cap E_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Then $\bigcup_{i=1}^{\infty} E_i$ is path connected.*

We have seen that $\mathbb{R} \setminus \mathbb{Q}$ is totally disconnected with respect to the usual topology (see Example 5.23). What about its counter part in $\mathbb{R}^2$? However, the set

$$E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}\}$$

is connected; in fact, path connected. Here are the details:

> Let $x, y \in E$ with $x \neq y$. Consider the straight line segment in $\mathbb{R}^2$ joining x and y and the straight line L perpendicular to it. for each $z \in L$, consider the path $f_z : [0,1] \to \mathbb{R}^2$ joining x to y consisting of the straight line segments joining x to z and z to y. Clearly, $\{f_z : z \in L\}$ is an uncountable family of paths joining x to y. Since $\mathbb{Q}$ is countable, there is at least one path, say f_{z_0} for some $z_o \in L$ such that its range lies entirely in E.

Thus, we have shown that for every pair of points $x, y \in E$, there exists a path in E joining x and y. In other words, E is path connected.

Chapter 6

Compact Space

The notion of a compact space is already familiar in the context of metric spaces, where it neatly captures the essence of finiteness in a geometric setting. However, this property can be generalized to the broader framework of general topological spaces. By abstracting compactness using open covers rather than metrics, we can preserve and extend many foundational results from real analysis and Euclidean geometry. This chapter explores how compactness serves as a powerful bridge, allowing us to generalize properties like boundedness and convergence to spaces that lack any notion of distance.

6.1 Definition and implications

Definition 6.1 Let X be a topological space and $E \subseteq X$. A collection $\mathcal{G}$ of subsets of X is called a **cover** of E if E is contained in the union of members of $\mathcal{G}$. A cover of E which consists of open sets in X is called an **open cover** of E. $\Diamond$

Definition 6.2 A topological space X is said to be a **compact space** if every open cover of X contains a finite sub-collection which is also a cover of X. $\Diamond$

Definition 6.3 A subset of a topological space is said to be **compact** if it is a compact space as a subspace. $\Diamond$

Theorem 6.4 *Let X be a topological space and $E \subseteq X$. Then E is a compact set iff for every family of open sets in X which covers E contains a finite sub-collection which also covers E.*

Proof. Suppose E is compact. Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of open sets in X which covers E, that is, $E \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$. Then

$$E = E \cap \Big(\bigcup_{\alpha\in\Lambda} G_\alpha \Big) = \bigcup_{\alpha\in\Lambda} (E \cap G_\alpha).$$

Thus, $\{E \cap G_\alpha : \alpha \in \Lambda\}$ be a family of open sets in E which covers E. Since E is compact, there exists $\alpha_1, \ldots, \alpha_n$ in Λ such that

$$E = \bigcup_{i=1}^{n} (E \cap G_{\alpha_i}).$$

Thus, $E \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus, we have proved that every family of open sets in X which covers E contains a finite sub-collection which also covers E.

Conversely, suppose every family of open sets in X which covers E contains a finite sub-collection which also covers E. We have to prove that E is compact. For this, let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of open sets in X which covers E, that is, $E = \bigcup_{\alpha \in \Lambda} V_\alpha$. Since each V_α is open in E, there exists open set G_α in X such that $V_\alpha = E \cap G_\alpha$. Thus, we have

$$E = \bigcup_{\alpha \in \Lambda} V_\alpha = \bigcup_{\alpha \in \Lambda} (E \cap G_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha.$$

By assumption, there exists $\alpha_1, \ldots, \alpha_n$ in Λ such that $E \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus,

$$E = E \cap \left(\bigcup_{i=1}^{n} G_{\alpha_i} \right) = \bigcup_{i=1}^{n} (E \cap G_{\alpha_i}) = \bigcup_{i=1}^{n} V_{\alpha_i}.$$

Thus, we have proved that every family of open sets in E which covers E contains a finite sub-collection which covers E, showing that E is compact. $\square$

The following can be verified easily (Exercise):

- Every finite subset of a topological space is compact.
- The set $\mathbb{Z}$ is not compact in $\mathbb{R}$ with respect to the usual topology.
- With respect to the usual topology on $\mathbb{R}$, the set $E = \{1/n : n \in \mathbb{N}\}$ is not compact, whereas the set $E \cup \{0\}$ is compact.
- With respect to the usual topology on $\mathbb{R}$, every open interval is non-compact.

Example 6.5 Every subset of a co-finite topological space is compact. Let X be a co-finite topological space and $A \subseteq X$. We know that A is closed iff A is a finite set. Let $\mathcal{G}$ be an open cover of A. Let $x_0 \in A$ and let $G_0 \in \mathcal{G}$ be such that $x_0 \in G_0$. If $A \subseteq G_0$, then there is nothing to prove. So, assume that

$A \not\subseteq G_0$. Then G_0^c is a finite set. Let $A \cap G_0^c = \{x_1, \ldots, x_k\}$ for some $k \in \mathbb{N}$. Let $G_1, \ldots, G_k$ in $\mathcal{G}$ be such that $x_j \in G_j$ for $j = 1, \ldots, k$ so that

$$A \cap G_0^c = \{x_1, \ldots, x_k\} \subseteq \bigcup_{j=1}^{k} G_j.$$

Then we have $A \subseteq G_0 \cup \bigcup_{j=1}^{k} G_i$. Thus, we have shown that $\mathcal{G}$ has a finite sub-collection which covers A. ◊

The above example shows that:

> In a topological space, every compact set need not be a closed set.

Look at the following examples too:

1. We have already seen in Example 6.5 that an infinite subset of a co-finite topological space is compact but not closed.
2. In every indiscrete space with more than one elements, singleton sets are compact, but not closed.
3. On the set $X = \{a, b, c\}$, consider the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b, c\}\}$. We observe that the set $\{b\}$ is compact, but not closed.
4. On the set $\mathbb{R}$, consider the topology $\mathcal{T} := \{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. With respect to this topology, singleton sets are compact, but not closed.
5. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$, let

$$B_r(x) = \{(y_1, y_2) \in \mathbb{R}^2 : |x_2 - y_2| < r\}.$$

   Let $\mathcal{T}$ be the family of all subsets A of $\mathbb{R}^2$ such that, for every $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$. Then, $\mathcal{T}$ is a topology on $\mathbb{R}^2$ (Verify). With respect to this topology, singleton sets are compact, but they are not closed.

The topological spaces in the above examples are not Hausdorff. This raises the following question:

- Are compact subsets of a Hausdorff space closed?

Then answer is in the affirmative, as the following theorem shows.

Theorem 6.6 *Every compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and K be a compact subset of X. To show that K is closed. That is to show that K^c is open. For this, it is enough to show that every point in K^c is an interior point. So, let $x \in K^c$. Since X is Hausdorff, for every $y \in K$, there exist open sets U_y, V_y such that

$$x \in U_y, \quad y \in V_y, \quad U_y \cap V_y = \varnothing.$$

Note that $\{V_y : y \in K\}$ is an open cover of K. Since K is compact, there exists $y_1, \ldots, y_n$ in K such that

$$K \subseteq V := \bigcup_{i=1}^{n} V_{y_i}.$$

Then, we have

$$x \in U := \bigcap_{i=1}^{n} U_{y_i}.$$

Since $U_{y_i} \cap V_{y_i} = \varnothing$, we have $U \cap V_{y_i} = \varnothing$. This is true for each $i = 1, \ldots, n$. Hence, $U \cap V = \varnothing$. This implies $U \cap K = \varnothing$ so that $U \subseteq K^c$. Thus, for each $x \in K^c$, we obtain an open set containing x and contained in K^c, showing that K^c is an open set. □

Corollary 6.7 *Every compact subset of a metrizable space is closed.*

While proving Theorem 6.6, we have actually proved the following theorem.

Theorem 6.8 *Let X be a Hausdorff space, K be a compact set and $x \in X$ such that $x \notin K$. Then there exist disjoint open sets containing x and K, respectively.*

Theorem 6.9 *If K_1 and K_2 are two disjoint compact sets in a Hausdorff space X, then there exist disjoint open sets containing K_1 and K_2, respectively.*

Proof. By Theorem 6.8, for each $x \in K_1$, there exist disjoint open sets U_x and V_x such that

$$x \in U_x, \quad K_2 \subseteq V_x.$$

$Then \{U_x : x \in K_1\}$ is an open cover of K_1. Since K_1 is compact, there exist $x_1, \ldots, x_n$ in K_1 such that

$$K_1 \subseteq U := \bigcup_{i=1}^{n} U_{x_i}.$$

Let $V = \bigcap_{i=1}^{n} V_{x_i}$. Then U and V are disjoint open sets containing K_1 and K_2, respectively. □

Theorem 6.10 *Every compact subset of a metric space is bounded.*

Proof. Let X be a metric space and E be a compact subset of X. For any given $r > 0$, we have

$$E \subseteq \bigcup_{x \in E} B(x, r).$$

Since E is compact, there exists $x_1, \ldots, x_n$ in E such that $E \subseteq \bigcup_{i=1} B(x_i, r)$. Hence, for any $x, y \in E$, there exists $k, \ell \in \{1, \ldots, n\}$ such that

$$x \in B(x_k, r), \quad y \in B(x_\ell, r).$$

Hence,

$$d(x, y) \le d(x, x_k) + d(x_k, x_\ell) + d(x_\ell, y) \le r + d(x_k, x_\ell) + r$$

Hence,

$$d(x, y) \le 2r + \max\{d(x_i, x_j) : i, j \in \{1, \ldots, n\}\}.$$

Hence, E is bounded. $\square$

By Corollary 6.7 and Theorem 6.10, every compact subset of a metric space is closed and bounded. However, a closed and bounded subset of a metric space need not be compact. A simple example is the discrete metric space on an infinite set. Here is another simple example.

Example 6.11 Let $X = (0, 2]$ be with usual metric. Then the set $E = (0, 1]$ is closed and bounded, but not compact. $\diamondsuit$

However, we have the following theorem.

Theorem 6.12 *Let X be a topological space and X_0 be a subspace of X. Let $E \subseteq X_0$. Then, E is compact in X_0 iff E is compact in X.*

Proof. Suppose E is compact in X_0. We show that E is compact in X. For this, let $\mathcal{G}$ be a family of open sets in X that covers E. Let

$$\mathcal{G}_0 := \{G \cap X_0 : G \in \mathcal{G}\}.$$

Then $\mathcal{G}_0$ is a family of open sets in X_0 that covers E. By hypothesis, there exists a finite sub-collection of $\mathcal{G}_0$ that covers E. This finite sub-collection is of the form $\{G_1 \cap X_0, \ldots, G_n \cap X_0\}$ for some $G_1, \ldots, G_n$ in $\mathcal{G}$. Hence,

$$E \subseteq \bigcup_{i=1}^{n} (G_i \cap X_0) \subseteq \bigcup_{i=1}^{n} G_i.$$

Thus, we obtained a finite sub-collection of $\mathcal{G}$ that covers E.

Conversely, suppose E is compact in X. We show that E is compact in X_0. For this, let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be a family of open sets in X_0 that covers E. We know that for each $\alpha \in \Lambda$, there exists an open set G_α in X such that $V_\alpha = G_\alpha \cap X_0$. Then, we see that, $\{G_\alpha : \alpha \in \Lambda\}$ is a family of open sets in X that covers E. Since E is compact in X, there exist $\alpha_1, \ldots, \alpha_n$ in Λ such that $E \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. Hence,

$$E \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap X_0) = \bigcup_{i=1}^{n} V_{\alpha_i}.$$

Thus, we have proved that the cover $\mathcal{V}$ consisting of open sets in X_0 has a finite sub-scollection that also covers E. Consequently, E is compact in X_0. □

Theorem 6.13 *Every closed subset of a compact space is compact.*

Proof. Let X be a compact topological space and C be a closed subset of X. Let $\mathcal{G}$ be an open cover of C. Since $G_0 := X \setminus C$ is open, the set $\{G_0\} \cup \{G : G \in \mathcal{G}\}$ is an open cover of X. Since X is compact, there exist $G_1, \ldots, G_n$ in $\mathcal{G}$ such that

$$X = G_0 \cup \Big(\bigcup_{i=1}^{n} G_i \Big).$$

Hence, $G_0^c \cap \Big(\bigcup_{i=1}^{n} G_i \Big)^c = \varnothing$, so that $C = G_0^c \subseteq \bigcup_{i=1}^{n} G_i$. Thus, the open cover $\mathcal{G}$ of C contains a finite sub-collection which also covers C, showing that C is compact. □

6.2 Continuity and compactness

Theorem 6.14 *Continuous image of a compact set is compact.*

Proof. Let X and Y be topological spaces, $f : X \to Y$ be a continuous function, and E be a compact set in X. To show that $f(E)$ is compact. For this, let $\mathcal{G}$ be an open cover of $f(E)$. Then we see that

$$\mathcal{A} = \{f^{-1}(G) : G \in \mathcal{G}\}$$

is an open cover on E. Since E is compact, there exists $G_1, \ldots, G_n$ in $\mathcal{G}$ such that $E \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i)$. This implies that $f(E) \subseteq \bigcup_{i=1}^{n} G_i$. Thus, the open cover $\mathcal{G}$ of $f(E)$ contains a finite sub-collection which also covers $f(E)$. Hence, $f(E)$ is compact. $\square$

Corollary 6.15 *Let X be a topological space and $\mathbb{R}$ be with usual topology. Let E be a compact subset of X and $f : X \to \mathbb{R}$ be a continuous function. Then, there exists $x_1, x_2 \in E$ such that*

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall\, x \in E.$$

Proof. Since X is compact, by Theorem 6.14, $f(E)$ is a compact subset of $\mathbb{R}$. In particular, $f(E)$ is closed and bounded, and hence there exists $u, v \in f(E)$ such that

$$u = \inf f(E), \quad v = \sup f(E).$$

Since $u, v \in f(E)$, there exists $x_1, x_2 \in E$ such that $f(x_1) = u$ and $f(x_2) = v$, which completes the proof. $\square$

We have seen examples of bijective continuous functions (see, e.g., Example 4.43 and Example 4.45) which are not homeomorphisms. However, if the domain space is compact and the co-domain space is Hausdorff, then such maps are homeomorphisms. More precisely, we hae the following theorem.

Theorem 6.16 *Every bijective continuous function from a compact space onto a Hausdorff space is a homeomorphism.*

Proof. Let X be a compact space and Y be a Hausdorff space, and let $f : X \to Y$ be a bijective continuous function. It is enough to show that f is a closed map. For this, let C be a closed set in X. Then, by Theorem 6.13, C is compact in X. Hence, by Theorem 6.14, $f(C)$ is compact. Since X is Hausdorff, by Theorem 6.6, $f(C)$ is closed in Y. This completes the proof. $\square$

An application of Theorem 6.16

Let X and Y be topological spaces and $F : X \to Y$ be a continuous function. Suppose for every $y \in Y$, there exists a unique $x \in X$ such that

$$F(x) = y. \tag{$*$}$$

Now, suppose (y_n) is a sequence in Y such that $y_n \to y$ for some $y \in Y$. For each $n \in \mathbb{N}$, let $x_n \in X$ be such that $F(x_n) = y_n$. One would like to know whether $x_n \to x$. Theorem 6.16 shows that the answer is in the affirmative if X is compact and Y is Hausdorff.

If X and Y are metric spaces with metrics d and ρ, respectively, with

X compact, and if $F : X \to Y$ is a continuous function, then the above consideration shows that small error in y can lead only lead to small error in the solution x. In other words, the solution of the equation $(*)$ depends continuously on the data y.

6.3 Finite intersection property

Theorem 6.17 (Finite intersection property) *Let $\mathcal{A} := \{K_\alpha : \alpha \in \Lambda\}$ be a family of non-empty compact subsets of a Hausdorff topological space X such that intersection of any finite subcollection of $\mathcal{A}$ is nonempty. Then $\bigcap_{\alpha\in\Lambda} K_\alpha$ is non-empty.*

Proof. Suppose $\bigcap_{\alpha\in\Lambda} K_\alpha = \varnothing$. Let $\alpha_0 \in \Lambda$ and $\Lambda_0 = \Lambda \setminus \{\alpha_0\}$. Then we have

$$K_{\alpha_0} \subseteq \Big(\bigcap_{\alpha\in\Lambda_0} K_\alpha\Big)^c = \bigcup_{\alpha\in\Lambda_0}^{\infty} K_\alpha^c.$$

Since X is Hausdorff, K_α is closed for every $\alpha \in \Lambda$, so that the above inclusion shows that $\{K_\alpha^c : \alpha \in \Lambda_0\}$ is an open cover of the compact set K_{α_0}. Hence, there exist $\alpha_1, \ldots, \alpha_N$ in Λ_0 such that

$$K_{\alpha_0} \subseteq \bigcup_{i=1}^{N} K_{\alpha_i}^c = \Big(\bigcap_{i=1}^{N} K_{\alpha_i}\Big)^c.$$

Hence, $K_{\alpha_0} \cap \Big(\bigcap_{i=1}^{N} K_{\alpha_i}\Big) = \varnothing$. This is a contradiction to the fact that intersection of any finite sub-collection of $\mathcal{A}$ is nonempty. $\square$

Corollary 6.18 *Let (K_n) be a sequence of non-empty compact subsets of a Hausdorff topological space X such that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^\infty K_n$ is non-empty.*

Proof. Suppose $\bigcap_{n=1}^\infty K_n = \varnothing$. This implies

$$K_1 \subseteq \Big(\bigcap_{n=2}^{\infty} K_n\Big)^c = \bigcup_{n=2}^{\infty} K_n^c.$$

Since X is Hausdorff, K_n is closed for every $n \in \mathbb{N}$ so that the above inclusion shows that $\{K_{n+1}^c : n \in \mathbb{N}\}$ is an open cover of the compact set K_1. Hence, there exists $N \in \mathbb{N}$ such that

$$K_1 \subseteq \bigcup_{n=2}^{N} K_n^c = \Big(\bigcap_{n=2}^{N} K_n\Big)^c.$$

Hence,

$$K_N = \bigcap_{n=1}^{N} K_n = K_1 \cap \Big(\bigcap_{n=2}^{N} K_n\Big) = \varnothing.$$

This is a contradiction to the fact that $K_n \neq \varnothing$ for all $n \in \mathbb{N}$. $\square$

Remark 6.19 We may recall from real analysis that if (F_n) is a sequence of non-empty closed sets in a complete metric space X such that $F_n \supseteq F_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$. $\Diamond$

6.4 Compact subsets of $\mathbb{R}$

We assume throughout this subsection that $\mathbb{R}$ is with usual topology.

Theorem 6.20 *For $a, b \in \mathbb{R}$ with $a < b$, the closed interval $[a, b]$ is compact.*

Proof. Let $\mathcal{G}$ be an open covering of $[a, b]$. Let

$$S = \{y \in [a, b] : [a, y] \text{ is covered by a finite subcollection of } \mathcal{G}\}.$$

Note that $a \in S$ and S is bounded above. Hence,

$$\beta := \sup S < \infty.$$

We first show that $\beta \in S$. For this, first observe that there exists $G \in \mathcal{G}$ such that $\beta \in G$. Hence, there exists $\alpha \in [a, \beta)$ such that $(\alpha, \beta] \subseteq G$. Therefore, by the definition of supremum, there exists $y \in S$ such that

$$\alpha < y \leq \beta.$$

Since

$$[a, \beta] = [a, y] \cup [y, \beta],$$

where $y \in S$ and $[y, \beta] \subseteq G$, it follows that $\beta \in S$.

Now, it is enough to show that $\beta = b$. To show this, assume the contrary that $\beta < b$. Let $G_0 \in \mathcal{G}$ be such that $\beta \in G_0$, and let $t \in [\beta, b)$ be such that $[\beta, t] \subseteq G_0$. Since

$$[a, t] = [a, \beta] \cup [\beta, t],$$

it follows that $t \in S$. This is a contradiction to the fact that $\beta = \sup S$. $\square$

Theorem 6.21 *Every closed and bounded subset of $\mathbb{R}$ is compact.*

Proof. Let E be a closed and bounded subset of $\mathbb{R}$. Since E is bounded, there exists $E \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Since $[a, b]$ is compact, and since closed subset of a compact set is compact, we obtain that E is compact. $\square$

We already know that every compact subset of a Hausdorff space is closed. Also, every compact subset of a metric space is bounded. Hence, we obtain the following.

Theorem 6.22 *A subset of E of $\mathbb{R}$ is compact with respect to the usual topology iff it is closed and bounded.*

Remark 6.23 We shall see that, a subset of E of $\mathbb{R}^n$ is compact (with respect to the usual topology) iff it is closed and bounded. $\Diamond$

6.5 Locally compact space

Definition 6.24 A topological space X is said to be a **locally compact space** if every point in X has an open neighbourhood contained in a compact subset of X. $\Diamond$

- A topological space X is locally compact iff for every $x \in X$, there exists open set V and a compact set K such that

$$x \in V \subseteq K.$$

Here is a characterization of locally compact spaces.

Theorem 6.25 *Let X be a Hausdorff space. The X is locally compact iff every point in X has an open neighbourhood whose closure is compact.*

Proof. Suppose X is locally compact and $x \in X$. Suppose V is an open set and K is a compact set such that $x \in V \subseteq K$. Since X is Hausdorff, K is closed so that $\bar{V} \subseteq K$. Being a closed subset of a compact set, $\bar{V}$ is compact.

The converse is obvious. $\square$

Example 6.26 The following statements can be easily verified (Exercise).

1. Every compact space is locally compact.

2. Every discrete space is locally compact.

3. With respect to the usual topology, the spaces $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ are locally compact, but not compact.

4. With respect to the usual topology, every nonempty open subset of $\mathbb{R}^n$ is locally compact, but not compact. $\Diamond$

Example 6.27 With respect to the usual topology, the spaces $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are neither compact nor locally compact. First let us show that $\mathbb{Q}$ is not locally compact. To see this, let $x \in \mathbb{Q}$. Suppose there exists an open set V in $\mathbb{Q}$ such that $x \in V$ and $\mathrm{cl}_{\mathbb{Q}}(V)$ is compact in $\mathbb{Q}$. Let G be an open set in $\mathbb{R}$ such that $V = G \cap \mathbb{Q}$, and let $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < b$ and

$$x \in (a, b) \cap \mathbb{Q} \subseteq G \cap \mathbb{Q} \subseteq V.$$

Note that $U := (a, b) \cap \mathbb{Q}$ is a closed subset of $\mathbb{Q}$ and $\mathrm{cl}_{\mathbb{Q}}(U) \subseteq \mathrm{cl}_{\mathbb{Q}}(V)$. Hence, $\mathrm{cl}_{\mathbb{Q}}(U) = U$ is compact in $\mathbb{Q}$. Hence, U is compact in $\mathbb{R}$ as well, which is not true, as compact subset of $\mathbb{R}$ are closed and bounded in $\mathbb{R}$, but U is not closed in $\mathbb{R}$. $\Diamond$

The conclusion in the above example also follows from the following general result.

Theorem 6.28 *Let X be a Hausdorff space and X_0 be a locally compact dense subspace of X. Then X_0 is open in X.*

Proof. Let $x \in X_0$. We show that there exists an open subset of X containing x and contained in X_0, that will complete the proof.

Since X_0 is locally compact, there exists a compact set $K \subseteq X_0$ and an open set V in X_0 such that

$$x \in V \subseteq K.$$

Since V is open in X_0, there exists an open set G in X such that $V = G \cap X_0$. We show that $G \subseteq X_0$. In fact, we show that $G \subseteq \overline{G \cap X_0}$ so that

$$G \subseteq \overline{G \cap X_0} = \bar{V} \subseteq \bar{K} = K,$$

where the last equality follows, since K is a compact subset of a Hausdorff space. For showing $G \subseteq \overline{G \cap X_0}$, let $y \in G$ and let H be an open subset of X containing y. Since $H \cap G$ is open set containing y and X_0 is dense in X, we have $H \cap G \cap X_0 \neq \varnothing$. That is, the open neighbourhood H of y contains some point from $G \cap X_0$. Hence, $y \in \overline{G \cap X_0}$. $\square$

- The dense subspaces $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ of $\mathbb{R}$ (with respect to the usual topology) are not open in $\mathbb{R}$, and hence are not locally compact.

6.6 One-point compactification

We know that the topological space $X = (0, 1]$ with usual metric is a locally compact Hausdorff space which is not compact. Now, if we add one point to it, we obtain a compact space $Y = [0, 1]$ with the usual topology. Note that X is dense in Y.

Given a locally compact Hausdorff space X which is not a compact space, can we obtain a compact space in which X is a dense subspace?

For example, consider the space $X = (0, 1)$, which is a locally compact Hausdorff space, which is not compact, with respect to the with usual topology. Let us consider the set

$$Y = X \cup \{p\}, \quad p := \{0, 1\}.$$

Consider the topology on Y for which the base consists of all open sub-intervals of $(0, 1)$ together with sets of the form $Y \setminus [a, b]$ for $a, b \in (0, 1)$. Then, it can be seen that, with this topology, Y is a compact Hausdorff space. Also, since every set of the form $Y \setminus [a, b]$ for $a, b \in (0, 1)$ contains points from X, X is a dense subspace of Y.

The above method can be adopted to obtain a compact Hausdorff space out of any locally compact Hausudorff space, as described in the following theorem.

Theorem 6.29 (One-point compactification) *Let $(X, \mathcal{T}_X)$ be a locally compact Hausdorff space which is not compact and let $Y = X \cup \{p\}$, where $p \notin X$. Let $\mathcal{T}_Y$ be the family of all those subsets A of Y such that either $A \in \mathcal{T}_X$ or $A = Y \setminus K$ for some compact subset $K \subseteq X$ or $A = Y$. Then, $\mathcal{T}_Y$ is a topology on Y such that*

(i) *$(Y, \mathcal{T}_Y)$ is a compact Hausdorff space and*

(ii) *X is a dense subspace of Y.*

Proof. It can be easily seen that $\mathcal{T}_Y$ is a topology on Y. To show that Y is compact with respect to this topology, let $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. If $Y = G_\alpha$ for some $\in \Lambda$, then, there is nothing to prove. So, assume that $G_\alpha \neq Y$ for all $\alpha \in \Lambda$. Let α_0 be such that $p \in G_{\alpha_0}$. Then $G_{\alpha_0} = Y \setminus K$ for some compact $K \subseteq X$ so that $Y \setminus G_{\alpha_0} = K$. Since K is also compact in Y, there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$. Thus,

$$Y = K \cup (Y \setminus K) \subseteq \Big(\bigcup_{i=1}^{n} G_{\alpha_i} \Big) \cup G_{\alpha_0},$$

showing that $(Y, \mathcal{T}_Y)$ is a compact space.

To see that Y is Hausdorff, it is enough to show that for each $x \in X$, there are disjoint open sets in Y containing x and p, respectively. Since X is locally compact, there exists a compact set $K \subseteq X$ and an open set $V \subseteq X$ such that $x \in V \subseteq K$. Then $Y \setminus K$ is an open set in Y containing p which is disjoint from V.

To show that X is dense in Y, it is enough to note that every open neighbourhood of p is either Y or it is of the form $Y \setminus K$ for some compact $K \subseteq X$. Since X is not compact, $(Y \setminus K) \cap X \neq \varnothing$. $\square$

Definition 6.30 Let $(X, \mathcal{T}_X)$ be a locally compact Hausdorff space which is not compact. Then the space Y obtained as in Theorem 6.29 is called the **one-point compactification** of X. $\Diamond$

We have already seen some examples of one-point compactification in the beginning of this section. Here are a few more. The reader is urged to work out the details.

Example 6.31 The one-point compactification of $X := \{1/n : n \in \mathbb{N}\}$ is homeomorphic with $X \cup \{0\}$. $\Diamond$

Example 6.32 The one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1. $\Diamond$

Example 6.33 The one-point compactification of $\mathbb{R}^2$ is homeomorphic with the unit sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. $\Diamond$

Example 6.34 The one-point compactification of $\mathbb{C}$ is homeomorphic with the Riemann sphere $\mathbb{C} \cup \{\infty\}$, obtained as in *stereographic projection.* $\Diamond$

Chapter 7

Product Topology and Quotient Topology

In this chapter, we shall define a topology on the Cartesian product of topological spaces using the topologies on the individual spaces in such a manner that the projections onto the factor spaces are continuous. We shall see that topological properties such as connectedness and compactness are preserved in the product space as well. Finally, we shall consider new spaces, called quotient spaces, formed by gluing together parts of the original spaces.

7.1 Product topology

7.1.1 Finite product of topological spaces

Let $\{(X_i, \mathcal{T}_i) : i = 1, \ldots, n\}$ be a family of non-empty topological spaces and and let X be the Cartesian product of $X_1, \ldots, X_n$, that is,

$$X = X_1 \times \cdots \times X_n,$$

which is also denoted by $\prod_{i=1}^n X_i$. We observe that the family

$$\mathcal{V} := \{V_1 \times \cdots \times V_n : V_i \text{ open in } \mathcal{T}_i,\ i = 1, \ldots, n\}$$

is closed under finite intersections. Hence, $\mathcal{V}$ is a base for a unique topology.

Definition 7.1 The topology for which

$$\mathcal{V} := \{V_1 \times \cdots \times V_n : V_i \text{ open in } \mathcal{T}_i,\ i = 1, \ldots, n\}$$

is a base is called the **product topology** on $\prod_{i=1}^n X_i$. $\Diamond$

- Let $X_1, \ldots, X_n$ be topological spaces. A subset A of $X := \prod_{i=1}^n X_i$ is open with respect to the product topology on X iff for every $x \in A$, there exists open sets V_i in X_i for $i = 1, \ldots, n$ such that $x \in V_1 \times \cdots \times V_n \subseteq A$.

The proof of the following theorem is obvious from the definition of a product topology.

Theorem 7.2 *Let $X_1, \ldots, X_n$ be non-empty topological spaces and X be their product space. Then, for each $i = 1, \ldots, n$, the map $\pi_i : X \to X_i$ defined by*

$$\pi_i(x) = x_i, \quad x := (x_1, \ldots, x_n) \in X,$$

is continuous and surjective, and the product topology on X is the weakest topology such that $\pi_1, \ldots \pi_n$ are continuous.

Proof. Let us show the conclusions one by one.

(1) Each π_i is surjective: Fix $i \in \{1, \ldots, n\}$ and let $y \in X_i$. For $j \in \{1, \ldots, n\}$, let $x_j \in X_j$ be such that $x_i = y$. Then

$$\pi_i(x_1, \ldots, x_n) = y.$$

(2) Each π_i is continuous: Fix $i \in \{1, \ldots, n\}$ and let G be open in X_i. Then

$$\pi_i^{-1}(G) = V_1 \times \cdots \times V_n,$$

where $V_i = G$ and $V_j = X_j$ for $j \neq i$.

(3) The product topology on X is the weakest topology such that $\pi_1, \ldots \pi_n$ are continuous: Let $\mathcal{T}$ be any topology on X such that $\pi_1, \ldots \pi_n$ are continuous. Then $\mathcal{T}$ must contain sets of the form $\pi_i^{-1}(G)$, where G open in X_i and $i \in \{1, \ldots, n\}$. Hence the product topology on X is weaker than $\mathcal{T}$. $\square$

Definition 7.3 The maps π_i in Theorem 7.2 are called **projection maps**. $\Diamond$

Example 7.4 Let $X_i = \mathbb{R}$ for $i = 1, \ldots, n$ with usual topology. Then the product topological space $X := \prod_{i=1}^n X_i$ is nothing but $\mathbb{R}^n$ with usual topology. For this topology, the set of all sets of the from $\prod_{i=1}^n (a_i, b_i)$ forms a base. $\Diamond$

7.1.2 Finite product of connected and compact spaces

Theorem 7.5 *Product of a finite number of connected spaces is connected.*

Proof. It is enough to show that the product of two connected spaces is connected.

Let X and Y be connected topological spaces. We show that $X \times Y$ is connected. For this, let us fix an element $(x_0, y_0) \in X \times Y$, and consider the subspaces $X \times \{y_0\}$ and and for each $x \in X$, the subspace $\{x\} \times Y$. Note that

1. $X \times Y = \bigcup_{x \in X} Z_x, \quad Z_x := (X \times \{y_0\}) \cup (\{x\} \times Y)$,
2. $(x, y_0) \in (X \times \{y_0\}) \cap (\{x\} \times Y) \quad \forall\, x \in X$,

3. $(x_0, y_0) \in Z_x$ for every $x \in X$.

Now, since $X \times \{y_0\}$ and $\{x\} \times Y$ are homeomorphic with the connected spaces X and Y, respectively, and (x, y_0) is a common point in these spaces, the subspace Z_x is conencted for each $x \in X$. Also, since $(x_0, y_0) \in Z_x$ for each $x \in X$, their union $X \times Y$ is connected. $\square$

For the proof of the next theorem, we shall make use of the following lemma, known as *tube-lemma*[1].

Lemma 7.6 (Tube lemma) *Let X and Y be topological spaces and $x_0 \in X$. If Y is compact and if A is an open subset of $X \times Y$ containing the slice $\{x_0\} \times Y$, then A contains the tube $G \times Y$, where G an open neighbourhood of x_0.*

Theorem 7.7 (Special case of Tychnoff's theorem) *Product of a finite number of compact spaces is compact.*

Proof. It it enough to prove for the case of $n = 2$. So, let X and Y be compact topological spaces, and $X \times Y$ be with product topology. Let $\mathcal{G}$ be an open cover of $X \times Y$. For each $x \in X$, since $\{x\} \times Y$ is compact, there exists $G_{x,1}, \ldots, G_k$ in $\mathcal{G}$ such that

$$\{x\} \times Y \subseteq \bigcup_{i=1}^{k} G_i.$$

By the tube-lemma, there exists a tube $V_x \times Y$ with V_x is an open neighbourhood of x such that

$$\{x\} \times Y \subseteq V_x \times Y \subseteq \bigcup_{i=1}^{k} G_i. \qquad (*)$$

Thus, $V_x \times Y$ is covered by a finite sub-collection of $\mathcal{G}$. Now, the space $\{V_x : x \in X\}$ is an open covering of X. Since X is compact, there exists $x_1, \ldots, x_\ell$ in X such that $X \subseteq \bigcup_{j=1}^{\ell} V_{x_j}$. Hence

$$X \times Y = \bigcup_{j=1}^{\ell} (V_{x_j} \times Y),$$

where each $V_{x_j} \times Y$ is covered by a finite number of members of $\mathcal{G}$, as per $(*)$.

Thus, we have proved that the open cover $\mathcal{G}$ of $X \times Y$ contains a sub-collection which also covers $X \times Y$. $\square$

[1]For its proof, see "Topology: A First Course" by J.R. Munkres

7.1.3 Infinite product of topological spaces

Next suppose we have a countably infinite collection of topological spaces, say $(X_i, \mathcal{T}_i)$ for $i \in \mathbb{N}$. Consider the Cartesian product $X := \displaystyle\prod_{i=1}^{\infty} X_i$.

- $\prod_{i=1}^{\infty} X_i$ is the set of all sequences of the form $(x_1, x_2, \ldots)$ with $x_n \in X_n$ for $n \in \mathbb{N}$. In other words, $\prod_{i=1}^{\infty} X_i$ is the set of all functions from $\mathbb{N}$ to $\bigcup_{i=1}^{\infty} X_i$.

For each $n \in \mathbb{N}$, let $\pi_n : \prod_{i=1}^{\infty} X_i \to X_n$ defined by

$$\pi_n(x) = x_n, \quad x := (x_1, x_2, \ldots).$$

Let

$$\mathcal{S} := \{\pi_n^{-1}(G) : G \in \mathcal{T}_n,\, n \in \mathbb{N}\}.$$

- The topology on $\prod_{i=1}^{\infty} X_i$ generated by $\mathcal{S}$ is the weakest topology such that each π_n is continuous.

Definition 7.8 The topology on $\prod_{i=1}^{\infty} X_i$ generated by

$$\mathcal{S} := \{\pi_n^{-1}(G) : G \in \mathcal{T}_n,\, n \in \mathbb{N}\}$$

is called the **product topology on $\prod_{i=1}^{\infty} X_i$**. $\Diamond$

- Let $\mathcal{V}$ be the collection of all finite intersections of members of $\mathcal{S}$. Then $\mathcal{V}$ is the family of all sets of the form $\prod_{i=1}^{\infty} V_i$, where each V_i is open in X_i and $V_i = X_i$ except for a finite number of i's, and this $\mathcal{V}$ is a base for the product topology on $\prod_{i=1}^{\infty} X_i$.

Definition 7.9 Let Λ be a non-empty set and for each $\alpha \in \Lambda$, let X_α be a non-empty set. Then the collection of all functions $f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for each $\alpha \in \Lambda$ is called the **Cartesan product** of X_α, $\alpha \in \Lambda$, and this Cartesian product is denoted by $\prod_{\alpha\in\Lambda} X_\alpha$. An element $f \in \prod_{\alpha\in\Lambda}^{\infty} V_\alpha$ is usually denoted by (x_α) or $(x_\alpha)_{\alpha\in\Lambda}$ with the understanding that $x_\alpha = f(\alpha)$ for $\alpha \in \Lambda$. $\Diamond$

Remark 7.10 The fact that the Cartesian product $\prod_{\alpha\in\Lambda} X_\alpha$ is nonempty is guaranteed by *axiom of choice*. $\Diamond$

Now, suppose that, for each $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is a topological space. For each $\beta \in \Lambda$, let $\pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$ be the function defined by

$$\pi_\beta(x) = x_\beta, \quad x := (x_\alpha)_{\alpha\in\Lambda}.$$

This π_β is called a **projection map** onto the *factor space* X_β.

Definition 7.11 Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces. Then the topology on $\prod_{\alpha\in\Lambda} X_\alpha$ generated by

$$\mathcal{S} := \{\pi_\beta^{-1}(G) : G \in \mathcal{T}_\beta,\ \beta \in \Lambda\}$$

is called the **product topology on** $\prod_{\alpha\in\Lambda} X_\alpha$. $\Diamond$

- The product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ is the weakest topology such that each π_β is continuous.

- $\mathcal{V}$ be the collection of all sets of the form $\prod_{\alpha\in\Lambda}^\infty V_\alpha$, where each V_α is open in X_α and $V_\alpha = X_\alpha$ except possible for a finite number of α's. Then $\mathcal{V}$ is a base for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

We state the following twp theorems. For their proof, the reader may refer the books given in the references.

Theorem 7.12 *If $\{X_\alpha : \alpha \in \Lambda\}$ is a family of connected spaces, then the product space $\prod\limits_{\alpha\in\Lambda} X_\alpha$ is connected.*

Theorem 7.13 (Tychonoff's theorem) *If $\{X_\alpha : \alpha \in \Lambda\}$ is a family of compact spaces, then the product space $\prod\limits_{\alpha\in\Lambda} X_\alpha$ is a compact space.*

We may define another topology on $\prod_{\alpha\in\Lambda} X_\alpha$ which seems more natural than the product topology, the so called *box topology.*

Suppose that, for each $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}_\alpha)$ is a topological space, and let $\mathcal{U}$ be the collection of all sets of the form $\prod_{\alpha\in\Lambda}^\infty V_\alpha$, where each V_α is open in X_α. Then $\mathcal{U}$ contains $\prod_{\alpha\in\Lambda}^\infty X_\alpha$ and it is closed under finite intersections, and hence $\mathcal{U}$ is a base for a unique topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

Definition 7.14 The topology on $\prod_{\alpha\in\Lambda} X_\alpha$ for which

$$\mathcal{U} := \Big\{ \prod_{\alpha\in\Lambda} V_\alpha : V_\alpha \in \mathcal{T}_\alpha \Big\}$$

is a base is called the **box topology** on $\prod_{\alpha\in\Lambda} X_\alpha$. $\Diamond$

- Box topology is stronger than the product topology.

- If Λ is a finite set, then product topology and box topology on the product set is the same.

- Box topology on the Cartesian product of an infinite number of connected spaces need not be connected (proof required!).

7.2 Quotient topology

7.2.1 Quotient map and quotient topology

Recall from Theorem 4.61 that, given a topological space $(X, \mathcal{T}_X)$, a set Y and a function $f : X \to Y$,

$$\mathcal{T}_Y := \{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\}$$

is a topology on Y, and it is the strongest topology on Y with respect to which f is continuous.

Definition 7.15 Let X and Y be topological spaces and $f : X \to Y$ be a surjective function . Then f is said to be a **quotient map** if the topology on Y is determined by f, and in that case, we say that the topology on Y is the **quotient topology** induced by f. $\Diamond$

Example 7.16 Let $X_1, \ldots, X_n$ be topological spaces and $X := \prod_{i=1}^n X_i$ be the corresponding product space. Then, for each $i = 1, \ldots, n$, the projection map $\pi_i : X \to X_i$ is a quaotient map. $\Diamond$

- A quotient map need not be open or closed, as the following examples show.

Example 7.17 Consider the function $f : [0, 2\pi) \to S^1$ defined by

$$f(x) = (\cos x, \sin x), \quad x \in [0, 2\pi).$$

Then it can be seen that the quotient topology on S^1 induced by f is the usual topology on it. Note that f is neither an open map nor a closed map. Indeed, $[0, \pi)$ is open in $[0, 2\pi)$ but not open in S^1, and $[\pi, 2\pi)$ is closed in $[0, 2\pi)$, but not closed in S^1. $\Diamond$

Example 7.18 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x$. Then f is a quotient map which is open but not a closed map. Clearly, image of every open set in $\mathbb{R}^2$ is open in $\mathbb{R}$, so that f is an open map. If we take $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$, then we see that A is a closed set in $\mathbb{R}^2$, but its image $\mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$, so that f is not a closed map. $\Diamond$

However, we have the following result.

Theorem 7.19 *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose $f : X \to Y$ is a surjective continuous function. If f is open or closed, then f is a quotient map.*

Proof. (i) Suppose f is an open map. We have to show that if $A \subseteq Y$, then $A \in \mathcal{T}_Y$ iff $f^{-1}(A) \in \mathcal{T}_X$. So let $A \subseteq Y$. Since f is continuous,

$$A \in \mathcal{T}_Y \quad \Longrightarrow \quad f^{-1}(A) \in \mathcal{T}_X.$$

Next, suppose $f^{-1}(A) \in \mathcal{T}_X$. Since f is an open map, $f(f^{-1}(A)) \in \mathcal{T}_Y$. Since f is surjective, we have[2] $A = f(f^{-1}(A))$. Thus, $A \in \mathcal{T}_Y$.

(ii) Suppose that f is a closed map. We have to show that if $A \subseteq Y$, then $A \in \mathcal{T}_Y$ iff $f^{-1}(A) \in \mathcal{T}_X$. So let $A \subseteq Y$. Again, since f is continuous,

$$A \in \mathcal{T}_Y \quad \Longrightarrow \quad f^{-1}(A) \in \mathcal{T}_X.$$

Now, let $f^{-1}(A) \in \mathcal{T}_X$. Then $f^{-1}(A^c) = [f^{-1}(A)]^c$ closed in X. Since f is a closed map, $f(f^{-1}(A^c))$ is closed in Y. Since f is surjective, $f(f^{-1}(A^c)) = A^c$. Thus, A^c closed, so that $A \in \mathcal{T}_Y$. $\square$

7.2.2 Quotient space

Let X be a set and $\sim$ be an *equivalence relation* on X. Let $\tilde{X} := X/\sim$ be the corresponding quotient set, that is, the family of all equivalence classes. Let $q : X \to \tilde{X}$ be the canonical map defined by

$$q(x) = [x], \quad x \in X,$$

where $[x]$ denotes the equivalence class of x. Then q is a surjective map.

Definition 7.20 Let X be a topological space and $\sim$ be an equivalence relation on X. Then the topological space $X/\sim$ with quotient topology induced by the canonical map $q : X \to X/\sim$, defined by

$$q(x) = [x], \quad x \in X,$$

is called the **quotient space** of X corresponding to the given equivalence relation $\sim$. $\diamondsuit$

Does every quotient map give rise to a quotient space?

The answer is in the affirmative, as the following consideration shows.

Let X and Y be a topological spaces, and $f : X \to Y$ be a quotient map. For $x, u \in X$, define

$$x \sim u \iff f(x) = f(u).$$

[2] $y \in A \iff y = f(x)$ for some $x \in f^{-1}(A) \iff y \in f(f^{-1}(A))$.

Then, it can be seen that, $\sim$ is an equivalence relation, which gives rise to the quotient space $X/\sim$. We show that Y is homeomorphic with $X/\sim$. For this purpose, we shall make use of the following lemma.

Lemma 7.21 *Let X, Y and Z be topological spaces, $f : X \to Y$ be a quotient map, $g : X \to Z$ be a continuous function, and $h : Y \to Z$ be such that $h \circ f = g$. Then h is continuous.*

Proof. Let $V \subseteq Z$. Since f is a quotient map and $h \circ f = g$, we have

$$h^{-1}(V) \text{ open in } Y \iff f^{-1}(h^{-1}(V)) \text{ open in } X \iff g^{-1}(V) \text{ open in } X.$$

Hence, using the fact that g is continuous, we obtain h is continuous. $\square$

Theorem 7.22 *Let X and Y be topological spaces and $f : X \to Y$ be a quotient map. Let $\sim$ be the equivalence relation on X induced by f and let the quotient set $\tilde{X} := X/\sim$ be endowed with the quotient topology induced by the canonical map $q : X \to \tilde{X}$. Then the function $\varphi : \tilde{X} \to Y$ defined by*

$$\varphi([x]) = f(x), \quad [x] \in \tilde{X},$$

is a bijective homeomorphism.

Proof. We observe that φ is bijective and

$$\varphi \circ q = f.$$

Hence, by Lemma 7.21, φ is continuous. Since

$$\varphi^{-1} \circ f = q,$$

again, by Lemma 7.21, φ^{-1} is continuous. $\square$

Example 7.23 Let $X = [0, 1]$ be with usual topology, and let

$$Y = \{0, 1\} \cup \{x : 0 < x < 1\}.$$

Define $f : X \to Y$ be defined by

$$f(x) = \begin{cases} \{0, 1\} & \text{if } x \in \{0, 1\}, \\ x & \text{if } 0 < x < 1. \end{cases}$$

Then f is a surjective function and it induces the quotient topology on Y. The quotient space Y is homeomorphic with the circle with usual topology. $\Diamond$

Example 7.24 Let $X = [0,1]$ be with usual topology. Consider the relation $\sim$ on X as

$$x \sim y \iff x = y \text{ or } \{x, y\} = \{0, 1\}.$$

This relation induces the quotient topology on $X/\sim$, which is homeomorphic with unit circle with usual topology. $\diamondsuit$

Example 7.25 Let $X = [0,1] \times [0,1]$ be with usual topology. Consider the relation $\sim$ on X as

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (a, y) \,\&\, (x', y')) = (b, y).$$

This relation induces the quotient topology on $X/\sim$, which is homeomorphic with a cylinder

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\, 0 \leq z \leq 1\}$$

with usual topology. $\diamondsuit$

Example 7.26 Let $X = [0,1] \times [0,1]$ be with usual topology. Consider the relation $\sim$ on X as

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } (x, y) = (a, y) \,\&\, (x', y')) = (b, 1 - y).$$

This relation induces the quotient topology on $X/\sim$, which is homeomorphic with a *Möbius strip*. $\diamondsuit$

Example 7.27 Consider the cylinder

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\, 0 \leq z \leq 1\}$$

with usual topology. Consider the relation $\sim$ on X as $(x, y, z) \sim (x', y', z')$ iff

$$(x, y, z) = (x', y', z') \text{ or } (x, y) = (x', y') \text{ with } x^2 + y^2 = 1\text{and}\, z \in \{0, 1\}.$$

This relation induces the quotient topology on $X/\sim$, which is homeomorphic with a *doughnut*. $\diamondsuit$

Chapter 8

Separation Axioms and Related Theorems

In our previous study of topological spaces, we observed that the Hausdorff property provides a fundamental baseline for geometric intuition. By ensuring that distinct points can be separated by disjoint open sets, the Hausdorff condition guarantees the uniqueness of limits for convergent sequences and closely mirrors the behavior of familiar metric spaces. However, the foundational fabric of topology allows for a much richer spectrum of isolation. A natural structural question arises when we expand our focus from individual points to broader topological configurations: What happens when we attempt to separate a point from a closed set (not containing it), or two entirely disjoint closed sets? In this chapter, we shall discuss how the hierarchy of separation axioms formally classifies these degrees of geometric isolation.

8.1 Separation axioms

Recall that, in a Hausdorff space, *any two distinct points can be separated by disjoint open sets*, that is, if x and y are distinct points in a Hausdorff space X, then there exist disjoint open sets U and V containing x and y, respectively.

We also know the following (see Theorem 6.9):

Theorem 8.1 *Let X be a Hausdorff space.*

(i) *If $x \in X$ and $K \subseteq X$ is compact such that $x \notin K$, then there exist disjoint open sets U and V containing x and K, respectively.*

(ii) *If K_1 and K_2 are disjoint compact sets in X, then there exist disjoint open sets U and V containing K_1 and K_2, respectively.*

There are topological spaces with *weaker assumptions of separation* and also *stronger assumptions of separation.*

Definition 8.2 Let X be a topological space. Then X is said to be a

1. T_0**-space** if for every pair (x, y) of distinct points in X, there exists an

open set G which contains at least one of x and y which does not contain the other point;

2. T_1**-space** if for every pair (x, y) of distinct points in X, there exist open sets U and V containing x and y, respectively, such that $x \notin V$ and $y \notin U$;

3. T_2**-space** if it is a Hausdorff space;

4. **regular space** if for every $x \in X$ and a closed set F in X which does not contain x, there exist disjoint open sets U and V containing x and F, respectively;

5. T_3**-space** if it is regular and T_1;

6. **normal space** if for every pair (F_1, F_2) of disjoint closed sets, there exist disjoint open sets U and V containing F_1 and F_2, respectively;

7. T_4**-space** if it is normal and T_1. $\Diamond$

Theorem 8.3 *Let X be a topological space. Then X is a T_1-space iff every singleton set in X is closed.*

Proof. Suppose X is a T_1-space. Then, for every $y \in X \setminus \{x\}$, there exists an open neighbourhood G of y such that $G \subseteq X \setminus \{x\}$. Hence, $X \setminus \{x\}$ is open.

Conversely, suppose every singleton set is closed. Then $X \setminus \{x\}$ and $X \setminus \{y\}$ are$y \notin X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. $\square$

We observe:

- $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$.
- Suppose that X is a topological space which is not a T_0-space. Then there exist disjoint points x, y in X such that every sequence which converges to x will also converge to y.
- Every compact Hausdorff space is T_4.

Remark 8.4 Let $\mathcal{S}$ be a family of subsets of a set X. In the following, we say that two disjoint subsets A and B be can be **separated by sets from $\mathcal{S}$**, if there exist disjoint sets $S_A, S_B \in \mathcal{S}$ such that $A \subseteq S_A$ and $B \subseteq S_B$. $\Diamond$

Example 8.5 Every metric space is T_0. $\Diamond$

Example 8.6 Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{\varnothing\} \cup \{\mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$.

1. $\mathcal{T}$ is T_0**, but not** T_1: For distinct $x, y \in X$, let $x < y$. Then $x \notin (y, \infty)$.
2. **Normal, but not regular**: One of any two disjoint closed sets is $\varnothing$, and hence $\varnothing$ and $\mathbb{R}$ are the disjoint open sets separating any two disjoint closed sets. It is not regular, because, $\mathbb{R}$ is the only open set which contains a nonempty closed set. $\Diamond$

Example 8.7 (T_0**, regular, normal, but not** T_1**)** An indiscrete space containing atleast two points is T_0, regular and normal, but not a T_1-space. $\Diamond$

Example 8.8 (T_2**, not regular)** Let $X = \mathbb{R}$, and let $\mathcal{T}_u$ be the usual topology on $\mathbb{R}$. Let $E_0 = \{1/n : n \in \mathbb{N}\}$, and let $\mathcal{T}$ be the topology generated by $\mathcal{T}_u$ and $\mathbb{R} \setminus E_0$. Then $\mathcal{T}$ is T_2, as $\mathcal{T}_u$ itself is T_2, but $\mathcal{T}$ is not regular, since 0 and E_0 cannot be separated by disjoint open sets in $\mathcal{T}$. Since $\{0\}$ is a closed set, $\mathcal{T}$ is not normal as well. $\Diamond$

Theorem 8.9 *Every metric space is* T_4.

Proof. Let (X, d) be a metric space and let $\mathcal{T}_d$ be the associated topology. Let A and B be disjoint closed sets in X. Let

$$f(x) = \text{dist}\,(x, A) - \text{dist}\,(x, B).$$

Then $f : X \to \mathbb{R}$ is continuous with

$$f(x) = \begin{cases} -\text{dist}\,(x, B), & x \in A, \\ \ \ \text{dist}\,(x, A), & x \in B. \end{cases}$$

Then A and B are contained in the disjoint open sets $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$, respectively. $\Box$

- Every metric space is T_k for $k \in \{0, 1, 2, 3, 4\}$.

Here is another characterization of Hausdorff spaces .

Theorem 8.10 *A topological space* X *is Hausdorff, that is,* T_2*, iff its diagonal set* $\Delta := \{(x, x) : x \in X\}$ *is closed in* $X \times X$.

Proof. Let X be Hausdorff. Let $(x, y) \in X \times X \setminus \Delta$. Then $x \neq y$ so that there exist disjoint open sets U and V containing x and y, respectively. Since $U \cap V = \varnothing$, we have $U \times V \subseteq X \times X \setminus \Delta$. This shows that $X \times X \setminus \Delta$ is open.

Conversely suppose that $X \times X \setminus \Delta$ is open. Let $(x, y) \in X \times X$ such that $x \neq y$. Since $X \times X \setminus \Delta$ is open, there exist disjoint open sets U and V in X such that $(x, y) \in U \times V \subseteq X \times X \setminus \Delta$. Thus, U and V are disjoint open sets containing x and y, respectively. $\Box$

Theorem 8.11 *Let X be a topological space.*

1. *X is regular iff for every $x \in X$ and for every open set U containing x, there exists an open set V such that $x \in V \subseteq \bar{V} \subseteq U$.*
2. *X is normal iff for every closed set F and for every open set U containing F, there exists an open set V such that $F \subseteq V \subseteq \bar{V} \subseteq U$.*

Theorem 8.12 *Let X be a topological space. Suppose for every closed set $F \subseteq X$ and a point $x \notin F$, there exist a function $f : X \to [0, 1]$ such that*

$$f(x) = 1, \quad f(y) = 0 \quad \forall\, y \in F.$$

Then X is a regular space.

Proof. Since $[0, 1]$ is Hausdorff, there exist disjoint open sets U and V in $[0, , 1]$ containing 1 and 0, respectively. Hence, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in X containing x and F, respectively. $\square$

Definition 8.13 A topological space X is said to be a **completely regular space** if for every closed set F and a point $x \notin F$, there exists a continuous function $f : X \to \mathbb{R}$ such that

$$f(x) = 1 \quad \text{and} \quad f(y) = 0 \quad \forall\, y \in F.$$

A completely regular space which is also T_1 is called a **Tychonoff space**. A completely regular space which is also Hausdorff is called a $T_{3\frac{1}{2}}$**-space**. $\Diamond$

It can be shown that

$$T_4 \implies T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1,$$

and the reverse implications need not hold.

8.2 Urysohn's lemma and Tietz extension theorem

First let us give a sufficient condition for normality of a space in terms of continuous functions.

Theorem 8.14 *Let X be a topological space. Suppose for every pair of disjoint closed sets A and B, there exist a continuous function $f : X \to [0, 1]$ such that*

$$f(x) = 1 \quad \forall\, x \in A \quad f(y) = 0 \quad \forall\, y \in F.$$

Then X is a normal space.

Proof. Since $[0,1]$ is Hausdorff, there exist disjoint open sets U and V in $[0,,1]$ containing 1 and 0, respectively. Hence, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in X containing A and B, respectively. $\square$

The converse of the above theorem is also true. The combined result is the so called *Urysohn's lemma.*

Theorem 8.15 (Urysohn's lemma) *A topological space X is normal iff for every pair of disjoint closed sets A and B, there exist a continuous function $f : X \to [0,1]$ such that*

$$f(x) = 1 \quad \forall\, x \in A \quad \textit{and} \quad f(y) = 0 \quad \forall\, y \in F.$$

Remark 8.16 Urysohn's lemma shows that complete regularity does not imply regularity. $\lozenge$

Recall from *pasting lemma* (Theorem (4.27) that if X_0 is a subspace of a topological space such that X_0 is open and closed, then for every topological space Y, every continuous function $f_0 : X_0 \to Y$ has a continuous extension to all of X.

If X is a normal space, then we have the conclusion in the pasting lemma for $Y = [0,1]$, requiring X_0 to be only a closed set.

Theorem 8.17 (Tietz extension theorem) *Let X be a normal space. Then for every closed set $X_0 \subseteq X$ and for every continuous function $f_0 : X_0 \to [0,1]$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = f_0(x)$ for all $x \in X_0$.*

Chapter 9

Problems

This chapter provides a comprehensive set of practice problems divided into six structured problem sheets. While the preceding text introduces the core theoretical frameworks and formulas, it deliberately leaves the deep practical execution to you. Mastering these concepts requires moving past passive reading and engaging in rigorous, independent problem-solving.

9.1 Problem sheet 1

1. Let $(X, \mathcal{T})$ be a topological space. If $A_i \in \mathcal{T}$ for $i = 1, \dots n$ for some $n \in \mathbb{N}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$. Deduce that finite union of closed sets is closed.

2. Let X be a topological space and $A \subseteq X$. Then A is open in X iff for every $x \in A$, there exists an open set G such that $x \in G \subseteq A$.

3. Let X be a metric space with metric d. Then

$$\mathcal{T} := \{A \subseteq X : \forall\, x \in A,\ \exists r > 0 \text{ such that } B_d(x, r) \subseteq A\}$$

   is a topology on X (called the *metric topology* on X).

4. Let (X, d) be a metric space. Then a subset A of X is open iff A is a union of open balls in X.

5. Show that the a topology on a nonempty set X is the discrete topology iff every singleton subset of X is open and, in that case, the topology is induced by the discrete metric on X, namely, the metric d defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

6. If X is any nonempty set, then $\mathcal{T} := 2^X$, the collection of all subsets of X, is a topology on X (called the *discrete topology* on X).

   With respect to this topology, every subset of X is open and closed.

7. If X is any nonempty set, then $\mathcal{T} := \{\varnothing, X\}$ is a topology on X, called the *indiscrete topology* on X.

   With respect to this topology, a proper nonempty subset of X is neither open or closed.

8. Show that the indiscrete topology on a nonempty set X is induced by a metric iff X is singleton, and in that case this topology is the discrete topology.

9. The indiscrete topology on a nonempty set X is Hausdorff iff $\#(X) = 1$, and in that case the topology is the discrete topology.

10. The indiscrete topology on a nonempty set X is induced by a metric iff X is singleton, and in that case this topology is the discrete topology.

11. If X is any nonempty set and $\mathcal{T}$ is a collection of subsets of X consisting of $\varnothing, X$ and those $A \subseteq X$ such that A^c is finite, then $\mathcal{T}$ is a topology on X (called the *co-finite topology* on X.)

    If X is an infinite set, then this topology is not Hausdorff.

12. The co-finite topology on a nonempty set X is induced by a metric iff X is a finite set, and in that case this topology is the discrete topology.

13. The co-finite topology on a nonempty set X is Hausdorff iff X is a finite set, and in that case the topology is the discrete topology.

14. If X is any nonempty set and $\mathcal{T}$ is a collection of subsets of X consisting of $\varnothing, X$ and those $A \subseteq X$ such that A^c is countable, then $\mathcal{T}$ is a topology on X (called the *co-countable topology* on X.)

    If X is an uncountable set, then this topology is not Hausdorff.

15. The co-countable topology on a nonempty set X is induced by a metric iff X is a countable set, and in that case this topology is the discrete topology.

16. The co-countable topology on a nonempty set X is Hausdorff iff X is a countable set, and in that case the topology is the discrete topology.

17. Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ such that $A \in \mathcal{T}$ iff for every $x \in A$, there exists $r > 0$ such that $[x, x + r) \subseteq A$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$ (called the *lower limit topology* on $\mathbb{R}$.)

18. Let $\mathcal{T}$ be the collection of all subsets of $\mathbb{R}$ such that $A \in \mathcal{T}$ iff for every $x \in A$, there exists $r > 0$ such that $(x - r, x] \subseteq A$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$ (called the *upper limit topology* on $\mathbb{R}$.)

19. Let $\mathbb{R}$ be with the topology induced by the usual metric. Then $[0,1)$ is neither open nor closed, and the sets $\varnothing$ and $\mathbb{R}$ are both open and closed.

20. Consider the metrics d_1, d_2, d_∞ on $\mathbb{R}^2$ defined by

$$\begin{aligned} d_1(x,y) &= |x_1 - y_1| + |x_2 - y_2|, \\ d_2(x,y) &= (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2}, \\ d_\infty(x,y) &= \max\{|x_1 - y_1|, |x_2 - y_2|\}, \end{aligned}$$

respectively, for $x, y \in \mathbb{R}^2$. The topologies induced by d_1, d_2, d_∞ are the same.

21. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, define $\rho(x,y) = |x_1 - y_1|$, and for $x \in X$ and $r > 0$, let $B_\rho(x,r) := \{u \in \mathbb{R}^2 : \rho(x,u) < r\}$. Then

$$\mathcal{T} := \{A \subseteq \mathbb{R}^2 : \forall\, x \in \mathbb{R}^2, \exists\, r > 0 \text{ such that } B_\rho(x,r) \subseteq A\}$$

is a topology on $\mathbb{R}^2$, which is not Hausdorff.

22. In a topological space, limit of a convergent sequence need not be unique.

23. In a discrete topological space every convergent sequence is eventually constant.

24. In Hausdorff space, limit of every convergent sequence is unique.

25. Lower limit topology and upper limit topology on $\mathbb{R}$ are Hausdorff.

26. The sequence $(-1/n)$ in $\mathbb{R}$ does not converge to any point with respect to the lower limit topology on $\mathbb{R}$.

9.2 Problem sheet 2

1. Let X be a nonempty set and let d and ρ be metrics on X. Suppose that for every $x \in X$ and for every $r > 0$, there exists $s > 0$ satisfying $B_\rho(x,s) \subseteq B_d(x,r)$. Show that $\mathcal{T}_d$ is weaker than $\mathcal{T}_\rho$.

2. Let d and ρ be metrics on a set X.

   (a) If there exists $c_1 > 0$ such that $d(x,y) \le c_1 \rho(x,y)$ for all $x, y \in X$, then $\mathcal{T}_d$ is weaker than $\mathcal{T}_\rho$.

   (b) If d and ρ are equivalent, that is, there exist $c_1 > 0$ and $c_2 > 0$ such that $c_2 \rho(x,y) \le d(x,y) \le c_1 \rho(x,y)$ for all $x, y \in X$, then $\mathcal{T}_d = \mathcal{T}_\rho$.

3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a nonempty set X such that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$.
   (a) If (x_n) is a sequence in X such that $x_n \to x$ with respect to $\mathcal{T}_2$, then $x_n \to x$ with respect to $\mathcal{T}_1$.
   (b) If $\mathcal{T}_1$ is strictly weaker than $\mathcal{T}_2$, then converse of (a) is not true.

4. On a set X, the discrete topology is the strongest topology, and the indiscrete topology is the weakest topology.

5. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on X, which is weaker than $\mathcal{T}_1$ and $\mathcal{T}_2$.

6. If $\{\mathcal{T}_\alpha : \alpha \in \Lambda\}$ is a family of all topologies on a set X, then $\bigcap_{\alpha\in\Lambda} \mathcal{T}_\alpha$ is a topology on X which is weaker than $\mathcal{T}_\alpha$ for every $\alpha \in \Lambda$.

7. Let $\mathcal{S}$ be a collection of subsets of X. Then the intersection of all topologies on X, each containing $\mathcal{S}$, is a topology on X, and it is the weakest topology containing $\mathcal{S}$.

8. Let $\mathcal{S}$ be the collection of all open intervals of the form (a, b) with $a < b$. Then any topology containing $\mathcal{S}$ contains the usual topology. In particular, the topology generated by $\mathcal{S}$ is the usual topology on $\mathbb{R}$.

9. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$, let
$$B_r(x) := \{y = (y_1, y_2) : |x_1 - y_1| < r\}.$$
   Let $\mathcal{T}$ be a collection of subsets of $\mathbb{R}^2$ such that $A \in \mathcal{T}$ iff for every $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$. Then
   (a) $\mathcal{T}$ is a topology on $\mathbb{R}^2$;
   (b) $\mathcal{T}$ is weaker than the usual topology, that is, the Euclidian topology;
   (c) $\mathcal{T}$ is not Hausdorff.

10. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$\varphi(x) = x_1, \quad x = (x_1, x_2) \in X.$$
    Then $\mathcal{T}_0 := \{\varphi^{-1}(G) : G \text{ open in } \mathbb{R}\}$ is a topology on $\mathbb{R}^2$, which is the same topology $\mathcal{T}$ in Problem 9.

11. Let $X = C[a, b]$, the space of all continuous real valued functions defined on $[a, b]$. Let $t_0 \in [a, b]$. For $x \in X$ and $r > 0$, let
$$B_r(x) := \{y \in C[a, b] : |y(t_0) - x(t_0)| < r\}.$$
    Let $\mathcal{T}$ be a family of subsets of X such that $A \in \mathcal{T}_0$ iff for every $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$. Then

(a) $\mathcal{T}$ is a topology on $C[a,b]$;

(b) $\mathcal{T}$ is weaker than the topology induced by the sup-metric;

(c) $\mathcal{T}$ is not Hausdorff;

(d) For a sequence (x_n) in X, $x_n \to x$ with respect to this topology $\mathcal{T}$ iff $x_n(t_0) \to x(t_0)$.

12. Let $X = C[a,b]$, the space of all continuous real valued functions defined on $[a,b]$. For $t_0 \in [a,b]$, let $\mathcal{T}_0 := \{\varphi^{-1}(G) : G \text{ open in } \mathbb{R}\}$. Then $\mathcal{T}_0$ is a topology on $C[a,b]$, which is same as the topology $\mathcal{T}$ in Problem 11.

13. Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}$ be a pseudp-metric[1] on X, that is,

(i) $\rho(x,y) \geq 0 \quad \forall\, x, y \in X$,

(ii) $\rho(x,y) = \rho(y,x) \quad \forall\, x, y \in X$,

(iii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y) \quad \forall x, y, z \in X$.

(Note that a pseudo-metric is a metric iff for $x, y \in X$, $\rho(x,y) = 0 \implies x = y$.) For $x \in X$ and $r > 0$, let $B_r(x) := \{y \in X : \rho(x,y) < r\}$. Then

(a) $\mathcal{T} := \{A \subseteq X : \text{ for every } x \in A,\ B_r(x) \subseteq A\}$ is a topology on X;

(b) $\mathcal{T}$ is Hausdorff iff ρ is a metric.

(Hint: If ρ is not a metric, then there exists $x \neq y$ such that $\rho(x,y) = 0$, so that for every $r > 0$, $y \in B_r(x)$.)

14. If X is with discrete topology, then for any $A \subseteq X$, $A^\circ = A = \bar{A}$.

15. If A is a subset of a topological space, then

$$A^\circ \subseteq A \subseteq \bar{A} = A \cup A' = A \cup \partial A = A^\circ \cup \partial A.$$

16. Let X be a topological space and $A \subseteq X$. Then

(a) $A^\circ = \bigcup\{G : G \subseteq A,\ G \text{ open in } X\}$.

(b) $\bar{A} = \bigcap\{F : F \supseteq A,\ F \text{ closed in } X\}$.

17. If Y is open in X, then a subset A of Y is open in Y iff A is open in X.

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[1] Eveery metric is a pseudo-metric, but the converse is not true. For example:

(i) The map $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\rho(x,y) = |x_1 - y_1|$ for $x, y \in \mathbb{R}^2$ defines a psedo-metric which is not a metric.

(ii) For $t_0 \in [a,b]$, the map $\rho : C[a,b] \times C[a,b] \to \mathbb{R}$ defined by $\rho(x,y) := |x(t_0) - y(t_0)|$ for $x, y \in C[a,b]$ defines a psedo-metric which is not a metric.

18. If Y is closed in X, then a subset A of Y is closed in Y iff A is closed in X.

19. Let X be a topological space and $Y \subseteq X$.

    (a) Every open set in Y is open in X iff Y is open in X.

    (b) An open set in Y can be open in X, without Y being open in X, as the following example shows.

    (c) Every subset of Y which is open in X is open in Y as well.

20. Let X be a topological space and $Y \subseteq X$. Then, every closed set in Y is closed in X iff Y is closed in X.

21. Let X be a topological space, Y is open in X and $A \subseteq Y \subseteq X$. Show that A is open in Y iff A is open X.

22. Let $X = \mathbb{R}$ be with usual topology and $Y = [0, 1]$.

    (a) $[0, 1]$ and $[0, 1/2)$ are open in Y but not open in X.

    (b) $(0, 1)$ is open in Y and open in X.

23. Every finite set in a Hausdorff space is closed.

9.3 Problem sheet 3

1. Let $\mathcal{S}$ be the collection of all intervals of the form (a, ∞) and $(-\infty, b)$. Then $\mathcal{S}$ is a sub-base for the usual topology on $\mathbb{R}$.

2. Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) with $a < b$. Then $\mathcal{B}$ is a base for the usual topology on $\mathbb{R}$.

3. Let $\mathcal{B}$ be the collection of all intervals of the form $[a, b)$ with $a < b$. Then $\mathcal{B}$ is a base for the lower limit topology on $\mathbb{R}$.

4. Let $\mathcal{B}$ be the collection of all intervals of the form $(a, b]$ with $a < b$. Then $\mathcal{B}$ is a base for the lower limit topology on $\mathbb{R}$.

5. Let $X = \{a, b, c\}$. Find a topology on X for which $\mathcal{B} := \{\varnothing, \{a, b\}, X\}$ is a base.

6. Every base is a sub-base. The converse is not true.

7. The topology itself is a base and sub-base.

8. If X is a metric space with metric d, then the collection of all open balls in X is a base for $\mathcal{T}_d$.

9. The space $\mathbb{R}^k$ with usual topology is second countable.

10. A discrete space X is second countable iff X is a countable set.

11. The discrete topology on $\mathbb{R} \setminus \mathbb{Q}$ is not seperable.

12. Let ℓ^∞ be the set of all bounded sequences of real numbers and let $d : \ell^\infty \times \ell^\infty \to \mathbb{R}$ be defined by
$$d(x, y) = \sup_{n\in\mathbb{N}} |x_n - y_n|, \quad x = (x_n), y = (y_n) \in \ell^\infty.$$
Then d is a metric on ℓ^∞ and the metric space (ℓ^∞, d) is not separable.

13. For a given $k \in \mathbb{N}$, let X_k be the set of all polynomials with rational coefficients. Let
$$d(p, q) = \sup_{0\leq t\leq 1} |p(t) - q(t)|, \quad p, q \in X_k.$$
Then d is a metric on X_k, and (X_k, d) is a separable space.

14. Let X be the set of all polynomials with real coefficients. Let
$$d(p, q) = \sup_{0\leq t\leq 1} |p(t) - q(t)|, \quad p, q \in X.$$
Then d is a metric on X, and (X, d) is a separable space.

15. Let $X : C([0, 1], \mathbb{R})$, the set of all continuous functions on $[0, 1]$ with real coefficients. Let
$$d(f, g) = \sup_{0\leq t\leq 1} |f(t) - g(t)|, \quad f, g \in X.$$
Then d is a metric on X, and (X, d) is a separable space.
(Hint: Use Weierstrass approximation theorem, which states that every continuous function defined on a closed and bounded interval is a uniform limit of a sequence of polynomials.)

16. Let (X, d) be a separable metric space. Then every subspace of X is separable.

17. The lower limit topology on $\mathbb{R}$ is first countable.

18. The co-countable topology on an uncountable space is not first countable

19. The co-finite topology on an infinite set is not first countable.

20. Let X be a topological space and $A \subseteq A$. If (x_n) is a sequence in A such that $x_n \to x$ for some $x \in X$, then $x \in \bar{A}$. But, its converse need not be true.

9.4 Problem sheet 4

1. If X is a discrete space and Y is any topological space, then every function $f : X \to Y$ is continuous.

2. Every continuous function is sequentially continuous; but its converse need not be true.

3. If X and Y are metrizable spaces, then a function $f : X \to Y$ is continuous iff it is sequentially continuous.

4. Let X be a co-countable topological space and Y be any topological space. Then every function $f : X \to Y$ is sequentially continuous.

5. Let X and Y be topological spaces, and $f : X \to Y$ be a bijective function. Then $f : X \to Y$ is a homeomorphism iff $f^{-1} : Y \to X$ is a homeomorphism

6. Let $\mathcal{X}$ be the family of all topological spaces. On $\mathcal{X}$, define the relation $\sim$ deefined by
$$X \sim Y \iff X \text{ is homeomorphic with } Y$$
is an equivalence relation (Reflexive, symmetric and transitive).

7. With usual topology on the following intervals, prove the following:
   (a) $(a, b) \sim (a, \infty) \sim (-\infty, \infty) \sim (-\infty, a) \sim (a, \infty) \sim (-\infty, a)$.
   (b) $(a, b] \sim [a, b) \sim [0, \infty) \sim (-\infty, a]$.
   (c) $(a, b) \not\sim [a, b], \quad (a, b) \not\sim [a, b), \quad [a, b] \not\sim [a, b), \quad (a, b) \not\sim [a, \infty)$.

8. Let X be a topological space and Y be a subset of X with subspace topology $\mathcal{T}_Y$. Then the inclusion map $f : Y \to X$, defined by $f(u) = u$, $u \in Y$, is
   (a) open iff Y is open in X;
   (b) closed iff Y is closed in X.

9. Let $\mathbb{R}$ and $\mathbb{R}^2$ be with usual topologies. Show that the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by
$$f(x) = (x, 0), \quad x \in \mathbb{R},$$
is a closed map, but not an open map.

10. Let X and Y be topological spaces, and $f : X \to Y$ be a bijective continuous function. Then the following are equivalent:

    (a) f is a homeomorphism.

    (b) f is an open map.

    (c) f is a closed map.

11. Find examples for the following:

    (a) An open map which is not a closed map.

    (b) A closed map which is not an open map.

    (c) A function which is both open and closed but not a homeomorphism.

    (d) A function which is continuous, open and closed but not a homeomorphism.

12. Verify the following.

    (a) A subset of a discrete space is connected iff it is a singleton set (Exercise).

    (b) If X is an uncountable set with co-countable topology, then X is connected.

    (c) If X is an infinite set with co-finite topology, then X is connected.

9.5 Problem sheet 5

1. Let X be a topological space and $E \subseteq X_0 \subseteq X$. Then E is a connected subset of X_0 iff E is a connected subset of X.

2. If $E_0, E_1, \ldots, E_n$ are connected subsets of a topological space X such that $E_{i-1} \cap E_i \neq \varnothing$ for $i = 1, \ldots, n$, then $\bigcup_{i=0}^{n} E_i$ is connected.

3. Suppose X is a disconnected topological space and (A, B) is a separation of X. If E is a connected subset of X, then either $E \subseteq A$ or $E \subseteq B$.

4. Let X be a topological space. Then X is connected iff every pair of points in X is contained in a connected subset of X.

5. Suppose (E_n) is a sequence of connected subsets of a topological space X such that $E_n \cap E_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} E_n$ connected.

6. (*Intermediate value theorem*) Let X be a connected topological space and let $f : X \to \mathbb{R}$ be a continuous function (with usual topology on $\mathbb{R}$). Suppose $x_1, x_2 \in X$ be such that $f(x_1) < f(x_2)$. Then, for every $\gamma \in \mathbb{R}$ such that
$$f(x_1) < \gamma < f(x_2),$$
there exists $x \in X$ such that $f(x) = \gamma$.

7. From the last theorem, deduce the following: Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $f(a) < f(b)$. If $\gamma \in \mathbb{R}$ is such that $f(a) < \gamma < f(b)$, then show that there exists $x \in [a, b]$ such that $f(x) = \gamma$.

8. Closure of a connected subset is connected.

9. Suppose A is a connected subset of a topological space X and $B \subseteq X$ is such that $A \subseteq B \subseteq \bar{A}$. Then B is connected.

10. A topological space is disconnected iff there exists a continuous function from X onto a two-point set with discrete topology.

11. Every discrete space is totally disconnected.

12. The topological spaces $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected with respect to the induced usual topology.

13. Let X be a topological space and $x, y, z \in X$. If there is a path from x to y and a path from y to z, then there is a path from x to z.

14. Let x, y be in a topological space X. If $f : [a, b] \to X$ is a continuous function such that $f(a) = x$ and $f(b) = y$, then for any closed interval $[c, d]$, there is a continuous function $g : [c, d] \to X$ such that $g(c) = x$ and $g(d) = y$.

15. A topological space X is path connected iff for every $x, y \in X$, there exists a continuous function $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

16. Every path connected space is connected.

17. Let A and B be path connected subsets of a topological space such that $A \cap B \neq \varnothing$. Then $A \cup B$ is path connected in X.

18. Let $\{E_\alpha : \alpha \in \Lambda\}$ be a family of path connected subsets of a topological space X such that $\bigcap_{\alpha \in \Lambda} E_\alpha \neq \varnothing$. Then $E := \bigcup_{\alpha \in \Lambda} E_\alpha$ is path connected.

19. Let $E_0, E_1, \ldots, E_n$ be path connected subsets of a topological space X such that $E_{i-1} \cap E_i \neq \varnothing$ for $i \in \{1, \ldots, n\}$. Then $\bigcup_{i=0}^{n} E_i$ is path connected.

20. Let (E_n) be a sequence of path connected subsets of a topological space X such that $E_n \cap E_{n+1} \neq \varnothing$ for every $n \in \mathbb{N}$. Then $\bigcup_{i=1}^{\infty} E_i$ is path connected.

9.6 Problem sheet 6

1. Justify the following statements:

   (a) Every finite subset of a topological space is compact.

   (b) The set $\mathbb{Z}$ is not compact in $\mathbb{R}$ with respect to the usual topology.

   (c) With respect to the usual topology on $\mathbb{R}$, the set $E = \{1/n : n \in \mathbb{N}\}$ is not compact, whereas the set $E \cup \{0\}$ is compact.

   (d) With respect to the usual topology on $\mathbb{R}$, every open interval is non-compact.

2. Justify the following statements:

   (a) In every indiscrete space with more than one elements, singleton sets are compact, but not closed.

   (b) On the set $X = \{a, b, c\}$, consider the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b, c\}\}$. We observe that the set $\{b\}$ is compact, but not closed.

   (c) On the set $\mathbb{R}$, consider the topology $\mathcal{T} := \{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. With respect to this topology, singleton sets are compact, but not closed.

   (d) For $x = (x_1, x_2) \in \mathbb{R}^2$ adn $r > 0$, let
   $$B_r(x) = \{(y_1, y_2) \in \mathbb{R}^2 : |x_2 - y_2| < r\}.$$
   Let $\mathcal{T}$ be the family of all subsets A of $\mathbb{R}^2$ such that, for every $x \in A$, there exists $r > 0$ such that $B_r(x) \subseteq A$. Then, $\mathcal{T}$ is a topology on $\mathbb{R}^2$ (Verify). With respect to this topology, singleton sets are compact, but they are not closed.

3. Let X be a Hausdorff space, K be a compact set and $x \in X$ such that $x \notin K$. Show that there exist disjoint open sets containing x and K, respectively.

4. If K_1 and K_2 are two disjoint compact sets in a Hausdorff space X, then there exist disjoint open sets containing K_1 and K_2, respectively.

5. Using the compactness of $[a, b]$ with respect to usual metric, deduce that every closed and bounded subset of $\mathbb{R}$ is compact.

6. Verify the following:

   (a) Every compact space is locally compact.

   (b) Every discrete space is locally compact.

   (c) With respect to the usual topology, the spaces $\mathbb{R}, \mathbb{Z}, \mathbb{N}$ are locally compact, but not compact.

   (d) With respect to the usual topology, every nonempty open subset of $\mathbb{R}^n$ is locally compact, but not compact. $\Diamond$

   (e) The subspaces $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ of $\mathbb{R}$ (with respect to the usual topology) are not locally compact.

7. What is the one-point compactification of the open interval $(0, 1)$ with respect to the usual topology? Why?

8. Let X_1 and X_2 be topological spaces and $X = X_1 \times X_2$. Verify the following:

   (a) $\mathcal{U} := \{V_1 \times V_2 : V_1 \text{ open in } X_1 \text{ and } V_2 \text{ open in } X_2\}$ is closed under finite intersections,

   (b) Let $\mathcal{T}$ be the topology on X for which $\mathcal{U}$ is a base. Then the functions $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ defined by

   $$\pi_1(x_1, x_2) = x_1, \quad \pi_2(x_1, x_2) = x_2 \quad \text{for } (x_1, x_2) \in X,$$

   are continuous and surjective,

   (c) $\mathcal{S} := \{V_1 \times X_2 : V_1 \text{ open in } X_1\} \cup \{X_1 \times V_2 : V_2 \text{ open in } X_2\}$ is a sub-base for the topology $\mathcal{T}$.

9. Let $X_1, \ldots, X_n$ be non-empty topological spaces and X be their cartesian product, and for each $i = 1, \ldots, n$, let $\pi_i : X \to X_i$ be defined by

   $$\pi_i(x) = x_i, \quad x := (x_1, \ldots, x_n) \in X.$$

   Show that the product topology on X is the weakest topology on X such that $\pi_1, \ldots \pi_n$ are continuous.

10. Let X be a topological space, Y be a set, and $f : X \to Y$. Verify the following.

    (a) $\mathcal{T}_Y = \{A \subseteq Y : f^{-1}(A) \text{ open in } X\}$ is a topology on Y,

(b) $f : X \to Y$ is continuous (with respect to the topolgy $\mathcal{T}_Y$ on Y),

(c) $\mathcal{T}_Y$ is the strongest topology on Y such that f is continuous.

11. Give examples to show that a quotient map need not be open or closed.

12. Let X and Y be topological space and $f : X \to Y$ be a quotient map. Let $\sim$ be the equivalence relation on X induced by f, i.e.,

$$x \sim u \iff f(x) = f(u).$$

Consider the quotient space $X/\sim$ induced by the canonical map $q : X \to X/\sim$ defined by

$$q(x) = [x], \quad x \in X.$$

Show that Y is homeomorphic with $X/\sim$.

13. Let X be a set, Y be a topological space, and $f : X \to Y$. Then

(a) $\mathcal{T}_X = \{f^{-1}(G) : G \text{ open in } Y\}$ is a topology on X,

(b) $f : X \to Y$ is continuous (with respect to the topology $\mathcal{T}_X$ on X),

(c) $\mathcal{T}_X$ is the weakest topology on X such that f is continuous.

14. Let X be a nonempty set, $\{(Y_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, and $f_\alpha : X \to Y_\alpha$ for $\alpha \in \Lambda$. Then the topology generated by

$$\mathcal{S} := \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \mathcal{T}_\alpha, \, \alpha \in \Lambda\}$$

is the weakest topology on X such that each f_α is continuous.

15. Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, Y be a nonempty set and $f_\alpha : X_\alpha \to Y$ for $\alpha \in \Lambda$. Then

$$\mathcal{T} := \{A \subseteq Y : f_\alpha^{-1}(A) \in \mathcal{T}_\alpha \text{ forall } \alpha \in X\}$$

is a topology on Y, and it is the strongest topology on Y such that each f_α is continuous.

Chapter 10

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Chapter 11

Appendix: Elements of Set Theory

Set theory is the grammar of mathematics. Just as structural grammar provides the rules to build meaningful sentences, set theory provides the structural foundation to build rigorous mathematical thoughts. It is almost impossible to master a modern subject such as *topology* without a basic knowledge of set operations, relations, and functions. Every advanced mathematical concept is built using the language of sets. This chapter introduces these essential tools, establishing the foundational vocabulary you will need to navigate the abstract landscapes of modern mathematics.

11.1 Union, intersection, and complements

Let S be a set and A and B are subsets of S. Then the **union** of A and B and the **intersection** of A and B are the sets

$$A \cup B := \{x \in S : x \in A \text{ or } x \in B\}, \quad A \cap B := \{x \in S : x \in A \text{ and } x \in B\},$$

respectively. More generally, let

$$\mathcal{A} := \{A_\alpha : \alpha \in \Lambda\}$$

be an **indexed family** of subsets of S, where Λ is called an **index set**. Then the union and intersection of all members of $\mathcal{A}$ are the sets

$$\bigcup_{\alpha \in \Lambda} A_\alpha := \{x \in S : x \in A_\alpha \text{ for some } \alpha \in \Lambda\},$$

$$\bigcap_{\alpha \in \Lambda} A_\alpha := \{x \in S : x \in A_\alpha \text{ for all } \alpha \in \Lambda\},$$

respectively. For sets A and B, the set $A \setminus B$ is defined as

$$\{x \in A : x \notin B\}.$$

If $A \subseteq S$, then $S \setminus A$ is called the **complement** of A in S. If the set S is understood from the context and $A \subseteq S$, then $S \setminus A$ is also denoted by A^c.

The following equalities, called **De Morgan's laws**,

$$\Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)^c = \bigcap_{\alpha\in\Lambda} A_\alpha^c, \quad \Big(\bigcap_{\alpha\in\Lambda} A_\alpha\Big)^c = \bigcup_{\alpha\in J} A_\alpha^c,$$

hold, where for $A \subseteq S$, A^c denotes the set

$$S \setminus A := \{x \in S : x \notin A\}.$$

Indeed,

$$x \in \Big(\bigcup_{\alpha\in\Lambda} A_\alpha\Big)^c \iff x \notin \bigcup_{\alpha\in\Lambda} A_\alpha \iff x \notin A_\alpha \quad \forall\, \alpha \in \Lambda \iff x \in \bigcap_{\alpha\in\Lambda} A_\alpha^c$$

and

$$x \in \Big(\bigcap_{\alpha\in\Lambda} A_j\Big)^c \iff x \notin \bigcap_{\alpha\in\Lambda} A_\alpha \iff x \in A_\alpha^c \text{ for some } \alpha \in \Lambda \iff \bigcup_{\alpha\in J} A_\alpha^c.$$

Also, for $A \subseteq S$,

$$A \cap \Big(\bigcup_{\alpha\in\Lambda} A_j\Big) = \bigcup_{\alpha\in\Lambda} (A \cap A_\alpha), \quad A \cup \Big(\bigcap_{\alpha\in\Lambda} A_\alpha\Big) = \bigcap_{\alpha\in\Lambda} (A \cup A_\alpha).$$

11.2 Cartesian product, relation and function

Given sets A and B, the **Cartesian product** of A and B is the set

$$A \times B := \{(x, y) : x \in A \text{ and } y \in B\}.$$

where (x, y) is called an **ordered pair**, in the sense that (x, y) and (x', y') are equal, and write $(x, y) = (x', y')$ iff $x = x'$ and $y = y'$.

Analogously, Cartesian product of any finite number of sets $A_1, \ldots, A_n$ is defined as

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_j \in A_j \text{ for all } j \in \{1, \ldots, n\}\},$$

where $(x_1, \ldots, x_n)$ is an **ordered n-tuples** in the sense that $(x_1, \ldots, x_n) = (x_1', \ldots, x_n')$ iff $x_j = x_j'$ for all $j \in \{1, \ldots, n\}$.

Any subset of $A \times B$ is called a **relation from A to B**. If R is a relation from A to B, that is $R \subseteq A \times B$ and if $(x, y) \in R$, the we write xRy. The set

$$\mathcal{D}(R) := \{x \in A : (x, y) \in R \text{ for some } y \in B\}$$

is called the **domain** of R and the set

$$\mathcal{R}(R) := \{y \in A : (x, y) \in R \text{ for some } x \in A\}$$

is called the **range** of R.

A relation R from A to B is called a **function** if the domain of R is A and for each x in the domain of R, there is one and only one y in the range of R such that $(x, y) \in R$. In other words, R is a function iff $\mathcal{D}(R) = A$ and

$$(x, y_1) \in R \text{ and } (x, y_2) \in R \quad \Longrightarrow \quad y_1 = y_2.$$

If R is a function from A to B, then we write $R : A \to B$, and we say that B is the **co-domain**, and if $(x, y) \in R$, then we write

$$y = R(x).$$

Thus,

$$\mathcal{R}(R) = \{R(x) : x \in A\}.$$

The collection of all functions from A to B is denoted by B^A.

Given a nonempty set S, a **sequence** in S is a function from $\mathbb{N}$ to S. Thus, $S^{\mathbb{N}}$ denotes the the set of all sequences in S.

A function $f : A \to B$ is said to be **one-one** or **injective** if

$$x_1, x_2 \in A \text{ with } x_1 \neq x_2 \quad \Longrightarrow \quad f(x_1) \neq f(x_2),$$

and it is said to be **onto** or **surjective** if its range is B. A function which is one and onto is said to be a **bijctive function**. Sets A and B are said to be in **one-one correspondence** if there is a bijection $\varphi : A \to B$, and in that case, there is a bijection $\psi : B \to A$ such that

$$\psi(\varphi(x)) = x, \quad \varphi(\psi(y)) = y \quad \forall\, x \in A,\, y \in B.$$

The function ψ (respectively, φ) is called the **inverse** of φ (respectively, ψ).

Now, let $f : A \to B$ and let $A_0 \subseteq A$ and $B_0 \subseteq B$. Then we write

$$f(A_0) = \{f(x) : x \in A_0\},$$
$$f^{-1}(B_0) = \{x \in A : f(x) \in B_0\}.$$

If $\{A_j : j \in J\}$ is an indexed family of subsets of A and $\{B_j : j \in J\}$ is an indexed family of subsets of B, then it can be seen that

$$f\Big(\bigcup_{j\in J} A_j\Big) = \bigcup_{j\in J} f(A_j), \quad f^{-1}\Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{j\in J} f^{-1}(B_j),$$

$$f^{-1}\Big(\bigcap_{j\in J} B_j\Big) = \bigcap_{j\in J} f^{-1}(B_j), \quad f\Big(\bigcap_{j\in J} A_j\Big) \subseteq \bigcap_{j\in J} f(A_j).$$

11.3 Cartesian product as family of functions

Given sets A and B, let $\mathcal{S}$ is the set of all functions $\varphi : \{1, 2\} \to A \cup B$ such that $\varphi(1) \in A$ and $\varphi(2) \in B$, and let $F : \mathcal{S} \to A \times B$ be defined by

$$F(\varphi) = (\varphi(1), \varphi(2)), \quad \varphi \in \mathcal{S}.$$

Then F is one-one and onto: Clearly, if $\varphi, \psi \in \mathcal{S}$ such that $F(\varphi) = F(\psi)$, then we have $(\varphi(1), \varphi(2)) = (\psi(1), \psi(2))$ so that $\varphi(1) = \psi(1)$ and $\varphi(2) = \psi(2)$. Hence, $\varphi = \psi$. Thus, F is one-one. Also, if $(x, y) \in A \times B$, then we may define $\varphi : \{1, 2\} \to A \cup B$ by $\varphi(1) = x$ and $\varphi(2) = y$, so that $\varphi \in \mathcal{S}$ and $F(\varphi) = (x, y)$.

In view of the above, we may consider $A \times B$ as the set of all functions from $\varphi : \{1, 2\} \to A \cup B$ such that $\varphi(1) \in A$ and $\varphi(2) \in B$.

Analogously, given any nonempty index set Λ and a family $\{A_\alpha : \alpha \in \Lambda\}$ of nonempty sets, we may define the **Cartesian product**

$$\prod_{\alpha \in \Lambda} A_\alpha$$

as the set of all functions $\varphi : \Lambda \to \bigcup_{\alpha \in \Lambda} A_\alpha$ such that $\varphi(\beta) \in A_\beta$ for all $\beta \in \Lambda$. Elements of $\prod_{\alpha \in \Lambda} A_\alpha$ are denoted by $(x_\alpha)_{\alpha \in \Lambda}$ with $x_\alpha \in A_\alpha$ for $\alpha \in \Lambda$.

11.4 Axiom of choice and Zorn's lemma

An important question in the context of the Cartesian product of an arbitrary family of sets is that whether $\prod_{\alpha \in \Lambda} A_\alpha$ is a nonempty set when every A_α is nonempty. The answer is given by an axiom in set theory, called the *axiom of choice*.

Axiom of choice: Given a nonempty index set Λ and a family $\{A_\alpha : \alpha \in \Lambda\}$ of nonempty sets, there exists a function $\varphi : \Lambda \to \bigcup_{\alpha \in \Lambda} A_\alpha$ such that $\varphi(\beta) \in A_\beta$ for all $\beta \in \Lambda$.

It is known that axiom of choice is equivalent to another statement in set theory, known as *Zorn's lemma*:

Zorn's lemma: Let $(S, \preceq)$ be a *partially ordered set*. If every *totally ordered subset* of of S has an *upper bound*, then the partially ordered set has a *maximal element*.

Let us define the terminologies used in the above.

- A relation $\preceq$ on a nonempty set S is called a **partial order** on S if it is reflexive, anti-symmetric, and transitive, that is,

  (i) $x \preceq x \quad \forall\, x \in S$ (reflexive),

  (ii) $x \preceq y,\, y \preceq x \implies x = y$ (antisymmetric),

  (iii) $x \preceq y,\, y \preceq z \implies x \preceq z$ (transitive).

- If $T \subseteq S$, then an element $b \in S$ is called an **upper bound** of T if $x \preceq b$ for all $x \in T$.

- A set $T \subseteq S$ is called **totally ordered** if for every $x, y \in T$, either $x \preceq y$ or $y \preceq x$.

- An element $s \in S$ is called a **maximal element** if for every $x \in S$, $s \preceq x \implies x = s$.

An illustration: Let $S = \{re^{i\theta} : 0 \leq r \leq 1,\, 0 \leq \theta < 2\pi\}$. For elements $x = r_1 e^{i\theta_1}$, $y = r_2 e^{i\theta_2}$ in S, define

$$x \preceq y \iff \theta_1 = \theta_2 \quad \& \quad r_1 \leq r_2.$$

Then $\preceq$ is a partial order on S. For each $\theta \in [0, 2\pi)$, the set

$$T_\theta := \{re^{i\theta} : 0 \leq r \leq 1\}$$

is a totally ordered set, and $e^{i\theta}$ is its upper bound. Every element of the form $e^{i\theta}$ is a maximal element of S.

If we define $S_0 = \{re^{i\theta} : 0 \leq r < 1,\, 0 \leq \theta < 2\pi\}$ with the partial order as above, then the set

$$T_{\theta,0} := \{re^{i\theta} : 0 \leq r < 1\}$$

is a totally ordered set in S_0 which does not have an upper bound, and S_0 does not have a maximal element.

- Zorn's lemma is used in mathematics at various contexts, for examples, for showing the existence of a *Hamel basis* for a vector space and for proving *Hahn-Banach extension thorem*.

It is also known that *Zorn's lemma* is equivalent to *continuum hypothesis*. In order to state continuum hypothesis, we need to recall the definition of a *cardinality* of a set.

11.5 Cardinality and continuum hypothesis

Let A and B be any two arbitrary sets. Then A and B are said to have the **same cardinality** if there is a bijection from A to B, and in that case, we write

$$\#(A) = \#(B).$$

Thus, given sets A and B, $\#(A) \neq \#(B)$ iff there is no one-one correspondence between A and B. For sets A and B, we write

- $\#(A) \leq \#(B)$ if there is an injective function from A into B, and
- $\#(A) < \#(B)$ if $\#(A) \leq \#(B)$, but $\#(A) \neq \#(B)$.

If $S = \varnothing$, then we write $\#(S) = 0$ and we denote

$$\#\{1, \ldots, n\} = n, \quad \#(\mathbb{N}) = \aleph_0, \quad \#(\mathbb{R}) = \mathbf{c}.$$

Let S be a nonempty set.

- S is said to be a **finite set** if $\#(S) = \#\{1, \ldots, n\}$ for some $n \in \mathbb{N}$, and in that case we write $\#(S) = n$, and say that cardinality of S is n.
- S is said to be a **countable set** if it is a finite set or if $\#(S) = \aleph_0$.
- S is said to be an **uncountable set** if $\#(S) = \mathbf{c}$.

We note that the function $f : (0, 1) \to (1, \infty)$ defined by

$$f(x) = \frac{1}{1 - x}, \quad 0 < x < 1,$$

is a bijection. Hence,

$$\#(0, 1) = \#(1, \infty).$$

It can be shown that $\#(1, \infty) = \#(\mathbb{R})$. Mote generally, it is true that if I is any interval, then

$$\#(I) = \#(\mathbb{R}).$$

We also have the following.

Theorem 11.1 *The cardinality of the set of all sequences with entries from* $\{0, 1\}$ *is* $\mathbf{c}$*, that is,*

$$\#(\{0, 1\}^{\mathbb{N}}) = \mathbf{c}.$$

This theorem can be established by defining a one-one correspondence between S and the open interval $I := \{x \in \mathbb{R} : 0 < x < 1\}$, by using the binary expansions of members of I.

We also observe that:

Theorem 11.2 *For any nonempty set S, the set $\{0,1\}^S$ is in one-one correspondence with the family of all subsets of S.*

Proof. Let $\widetilde{S}$ be the family of all subsets of S. Then we see that the $\varphi : \widetilde{S} \to \{0,1\}^S$ defined by $\varphi(A) = \chi_A$, the characteristic function of A, is a cone-one correspondence. $\square$

- In view of the above theorem, the **power set** of S, namely, the family of all subsets of S is denoted by 2^S.

Thus, we have shown that

$$\#(\{0,1\}^S) = \#(2^S).$$

Thus, we have proved that

Theorem 11.3

$$\#(\mathbb{R}) = \#(2^{\mathbb{N}}).$$

We note that

$$n < \aleph_0 \leq \mathbf{c} \quad \forall\, n \in \mathbb{N}.$$

One may ask whether we have $\aleph_0 < \mathbf{c}$. This is the case, in view of the following theorem.

Theorem 11.4 *For any nonempty set S, $\#(S) < \#(2^S)$. In particular,*

$$\aleph_0 = \#(\mathbb{N}) < \#(2^{\mathbb{N}}) = \mathbf{c}.$$

Proof. We have to prove that there is an injective function from S to 2^S, but there is no surjective function from S onto 2^S.

Clearly, the map $x \mapsto \{x\}$ is an injective function from S to 2^S. Consider any function $\varphi : S \to 2^S$, and consider the set

$$A := \{x \in S : x \notin \varphi(x)\}.$$

Then $A \in 2^S$. We show that $A \notin R(\varphi)$. Assume on the contrary that $A \in R(\varphi)$, and let $x_0 \in S$ such that $\varphi(x_0) = A$. Then any one of the following alternatives must hold:

$$x_0 \in A \quad \text{or else} \quad x_0 \notin A.$$

But, we have

$$x_0 \in A \quad \Longrightarrow \quad x_0 \notin \varphi(x_0) = A;$$

and

$$x_0 \notin A \quad \Longrightarrow \quad x_0 \in \varphi(x_0) = A.$$

Thus, each of the above two alternatives leads to a contradiction. This shows that, there is no $x_0 \in S$ such that $\varphi(x_0) = A$. $\square$

For nonempty sets A and B, let us denote $\#(B^A) := (\#B)^{\#A}$, that is, if we denote $\alpha := \#(A)$ and $\beta := \#(B)$,

$$\#(B^A) := \beta^\alpha.$$

Thus, for any nonempty set S, if we denote $s := \#(S)$, we have

$$s < 2^s,$$

and if we write $s_n := 2^{s_{n-1}}$ with $s_0 = s$, then we have

$$s := s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} < \cdots .$$

In particular, taking $S = \mathbb{N}$ and denoting $c_0 := \aleph_0$, $c_1 = \mathbf{c}$, and defining $c_n := 2^{c_{n-1}}$ we have

$$\aleph_0 < \mathbf{c} < c_2 < \cdots < c_n < c_{n+1} < \cdots .$$

Having proved

$$\aleph_0 < \mathbf{c} = \#(2^{\mathbb{N}})$$

one may ask whether there is a cardinality which is strictly greater than $\aleph_0$ and strictly less than $\mathbf{c}$. The negation of this is the so called *continuum hypothesis*, which is, in fact, an axiom in set theory.

Continuum hypothesis: There does not exist a set S such that

$$\aleph_0 < \#(S) < \mathbf{c}.$$

Remark 11.5 It is known that Axiom of choice, Zorn's lemma, and Continuum hypothesis are equivalent. However, none of them can be proved or disproved using the standard rules of set theory! This is a consequence of the work of *Kurt Gödel* and *Paul Cohen* in the last century, which is considered to be the most important work in the foundations of mathematics. $\Diamond$