Unstacking, Conway's soldiers and Fibonacci numbers

Mathematics Department Colloquium

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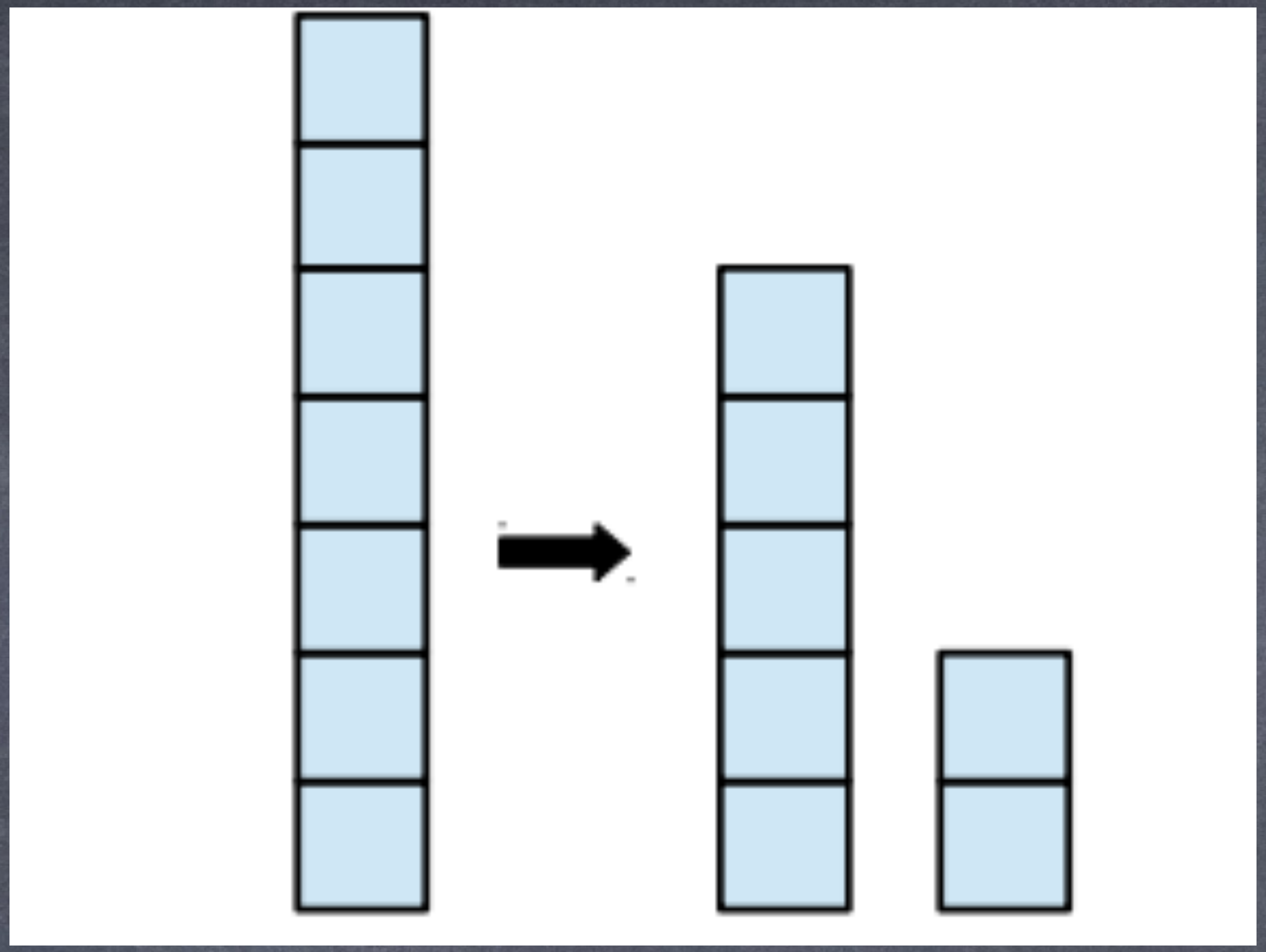
THE UNSTACKING GAME

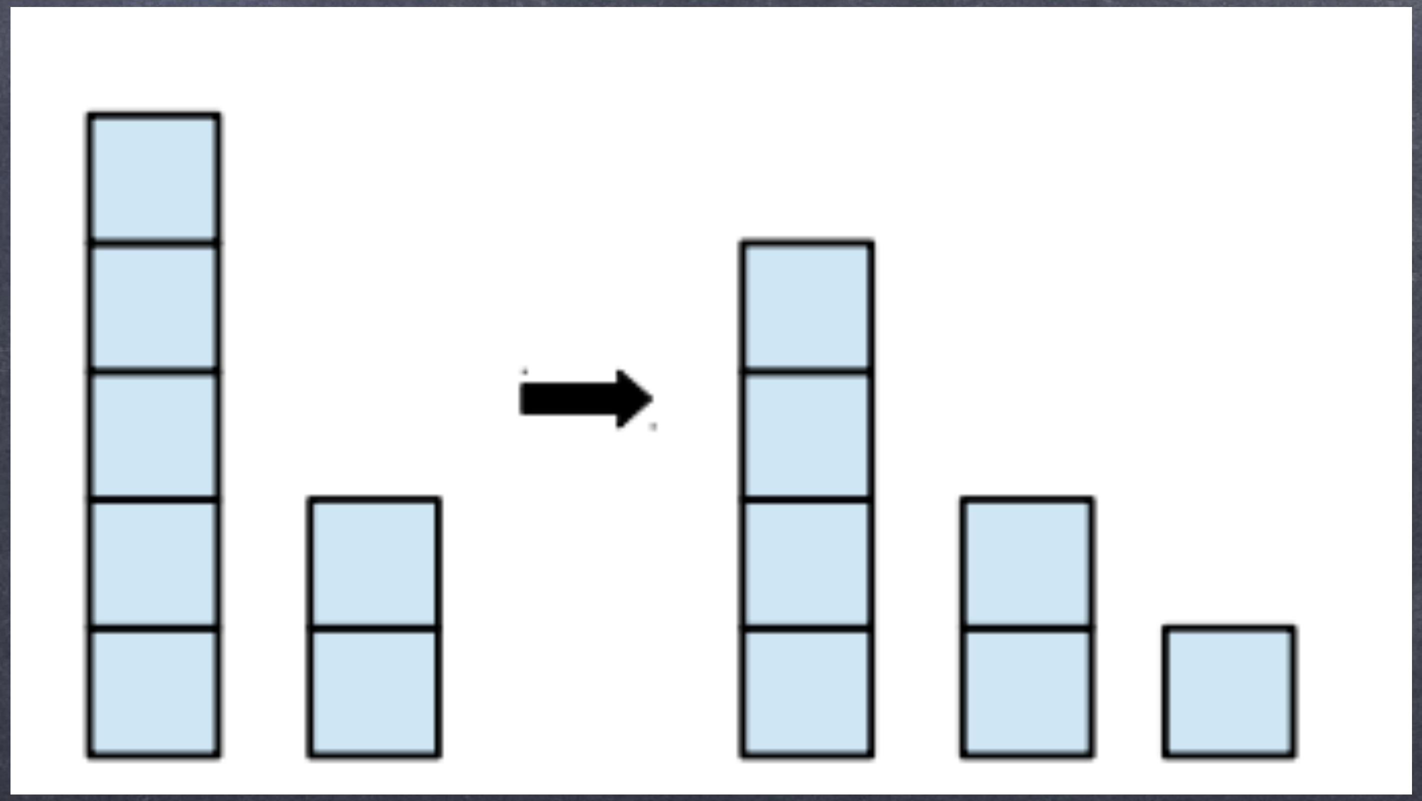
Given $(n+1)$ bricks stacked one on top of the other, split them into a set of two piles, one of them consisting of n_1 bricks and the other n_2 bricks such that $n_1 + n_2 = n + 1$. Suppose $n = 6$, then this first step may look like this:

You get $5 \times 2 = 10$ points for this first step



Now, you continue the game by splitting one of the two piles again, say the one with 5 bricks into two piles of 4 bricks and 1 brick:

This time, you earn $4 \times 1 = 4$ points.



The game ends when each pile has only one brick left.

The number of points you earn is the total of the points you have earned at each step.

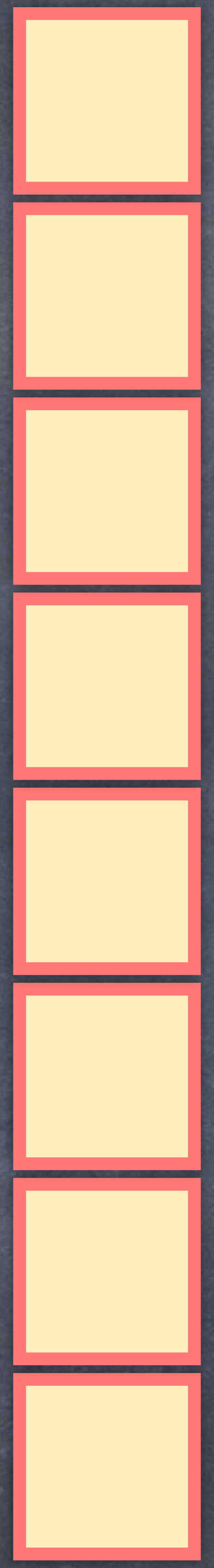
Goal:

Given a n, find a strategy to maximise the number of points earned.

What follows is an example with $n = 7$.

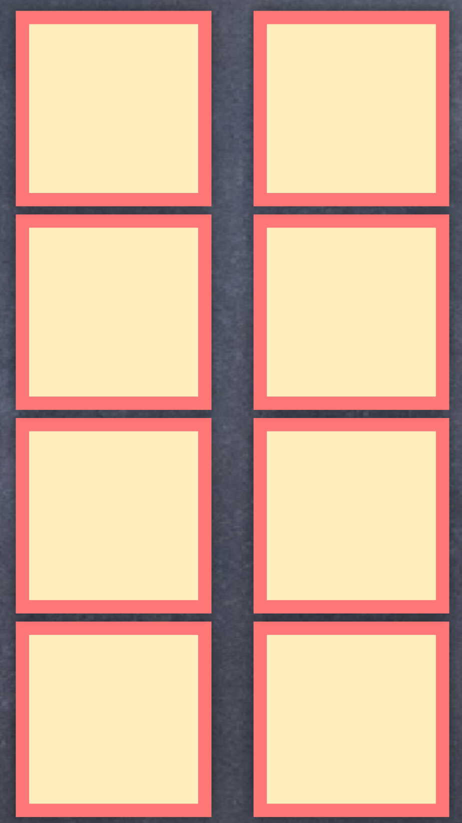
SCORE

0



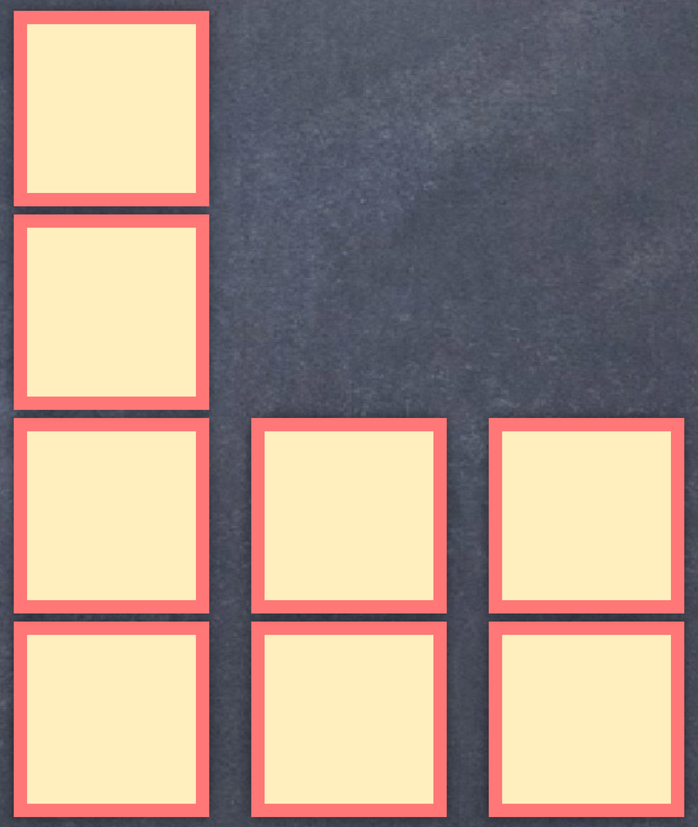
SCORE

16



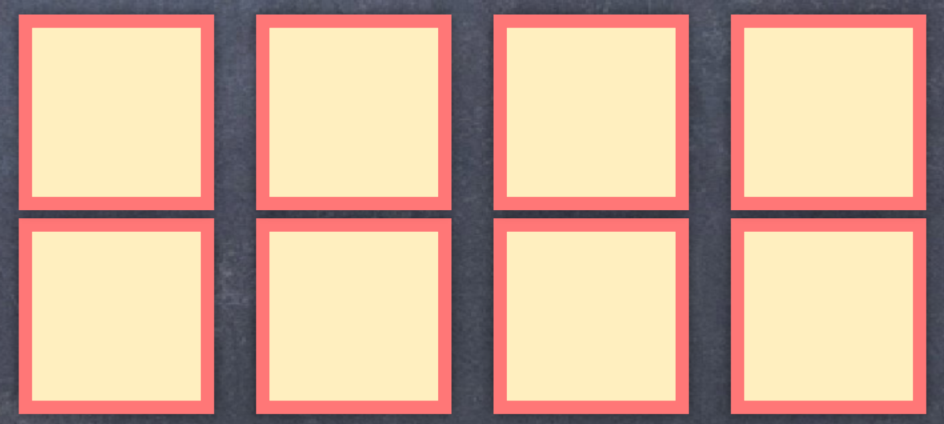
SCORE

16 + 4



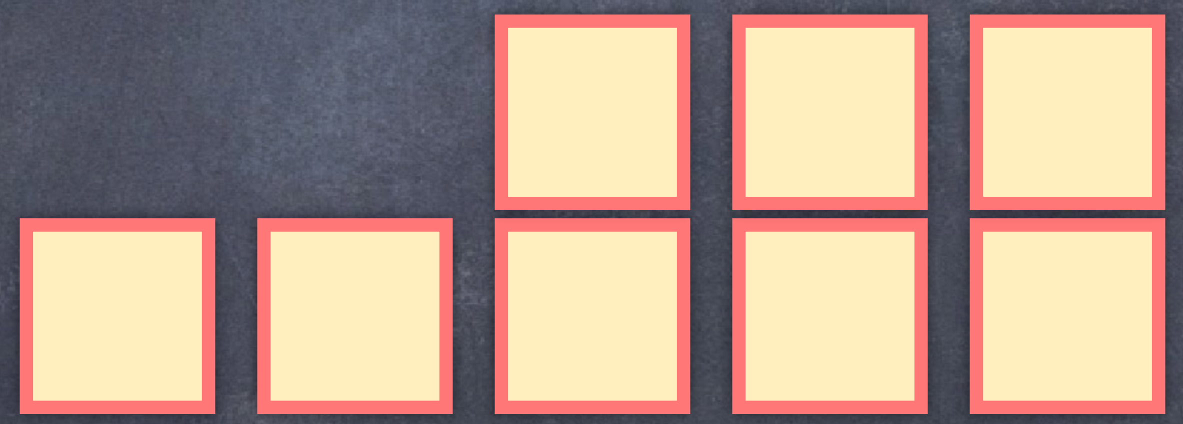
SCORE

16 + 4 + 4



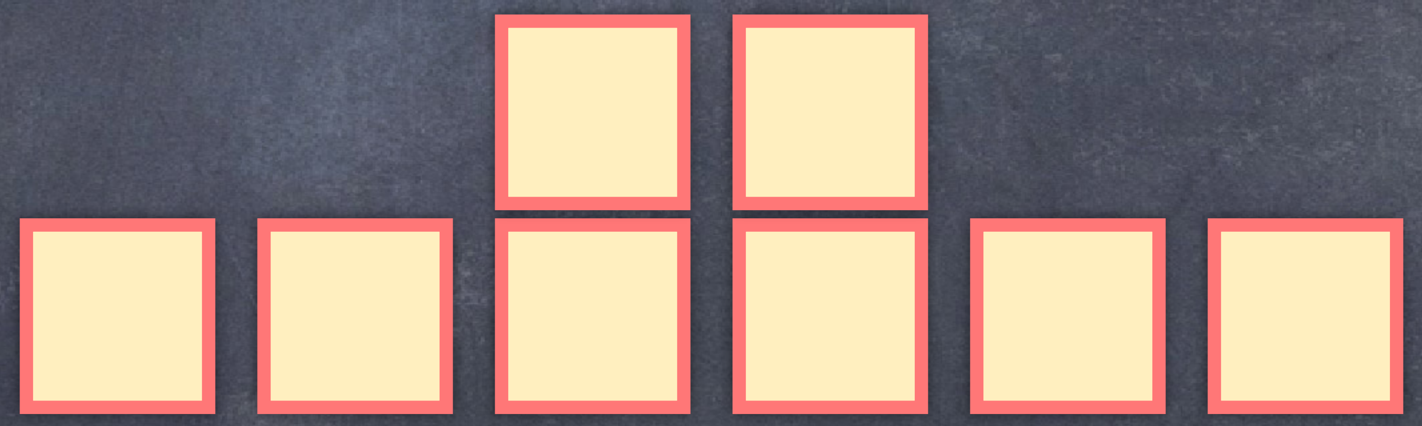
SCORE

16 + 4 + 4 + 1



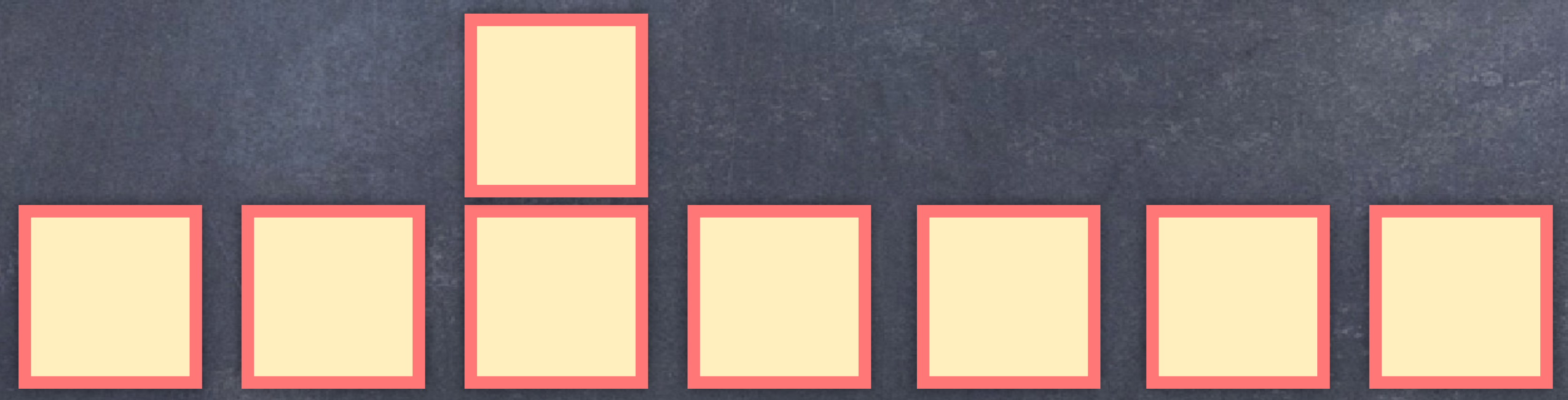
SCORE

16 + 4 + 4 + 1 + 1



SCORE

16 + 4 + 4 + 1 + 1 + 1



SCORE

16 + 4 + 4 + 1 + 1 + 1 + 1

SCORE

28

Big Question: Can you do better? Can you do any worse?

A different way you could play the game with any n is the following.

$n+1 \to 1, n,$	Points earned: n
$n \to 1, n-1,$	Points earned: $n-1$
$n-1 \to 1, n-2,$	Points earned: $n-2$
$\vdots$	$\vdots$

Continuing in this manner, after n steps, we will have nothing to split.

The points we would have earned in the bargain is $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Theorem: No matter what strategy is used, the score for the unstacking game with $n+1$ blocks is $\frac{n(n+1)}{2}$.

The proof is by the principle of strong induction:

Let $P(n)$ be a property that applies to natural numbers. Suppose that the following are true:

$P(0)$ is true. For any k, if $k \in \mathbb{N}$, $P(0), P(1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

Then for any $n \in \mathbb{N}$, $P(n)$ is true.

The proof

For our base case, we prove $P(0)$, that any strategy for the unstacking game with one block will always yield $\frac{0(0+1)}{2} = 0$ points.

This is true because the game immediately ends if the only stack has size one, so all strategies immediately yield 0 points.

For the inductive hypothesis, assume that for some $n \in \mathbb{N}$ and all $k \in \mathbb{N}, k \leq n$, $P(k)$ holds.

Under this hypothesis, to show that P(n+1) holds.

Finishing the proof

Since each stack must have at least one block in it, this means that $k \geq 0$ (so that $k+1 \geq 1$) and that $k \leq n$ (so that $(n-k)+1 \geq 1$).

Consequently, we know that $0 \leq k \leq n$, and by the inductive hypothesis we have that the total number of points earned from splitting the stack of $(k+1)$ blocks down must be $\frac{k(k+1)}{2}$.

Similarly, since $0 \leq n-k \leq n$, again by the inductive hypothesis, the total score for the stack of $(n-k)+1$ blocks must be $\frac{(n-k)(n-k+1)}{2}$.

The last step in the proof

Let us consider the total score for this game. The initial move yields $(k+1)(n-k+1)$ points.

The two subgames yield $\frac{k(k+1)}{2}$ and $\frac{(n-k+1)(n-k)}{2}$ points, respectively.

This means that the total number of points earned is

$$(k+1)(n-k+1) \; + \frac{(n-k+1)(n-k)}{2} \; + \; \frac{k(k+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

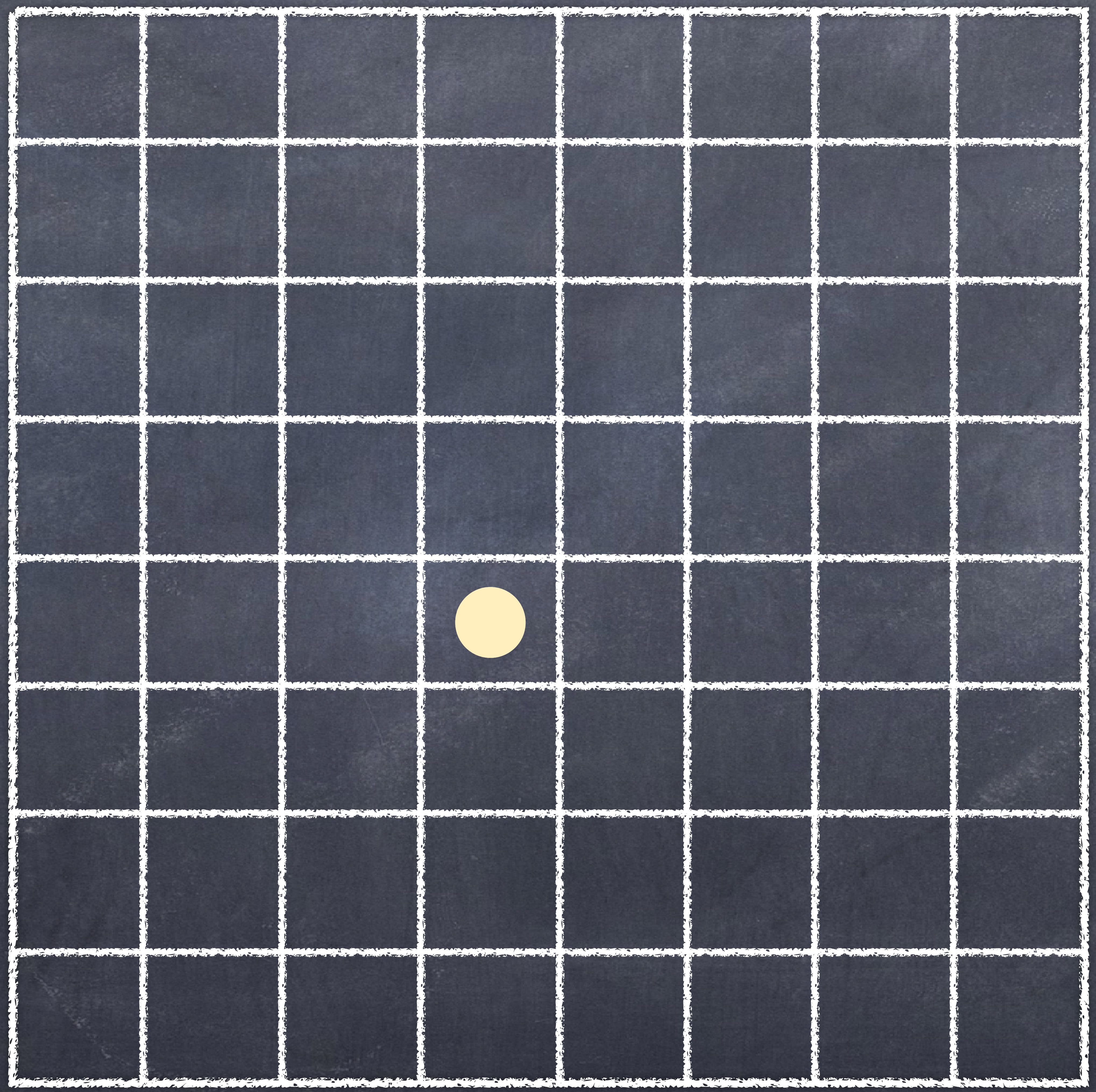
This is the answer with a stack of size $n+2$. The inductive step is therefore verified, completing the proof.

Freeing the aliens

In one move...

Alien at position (x,y)

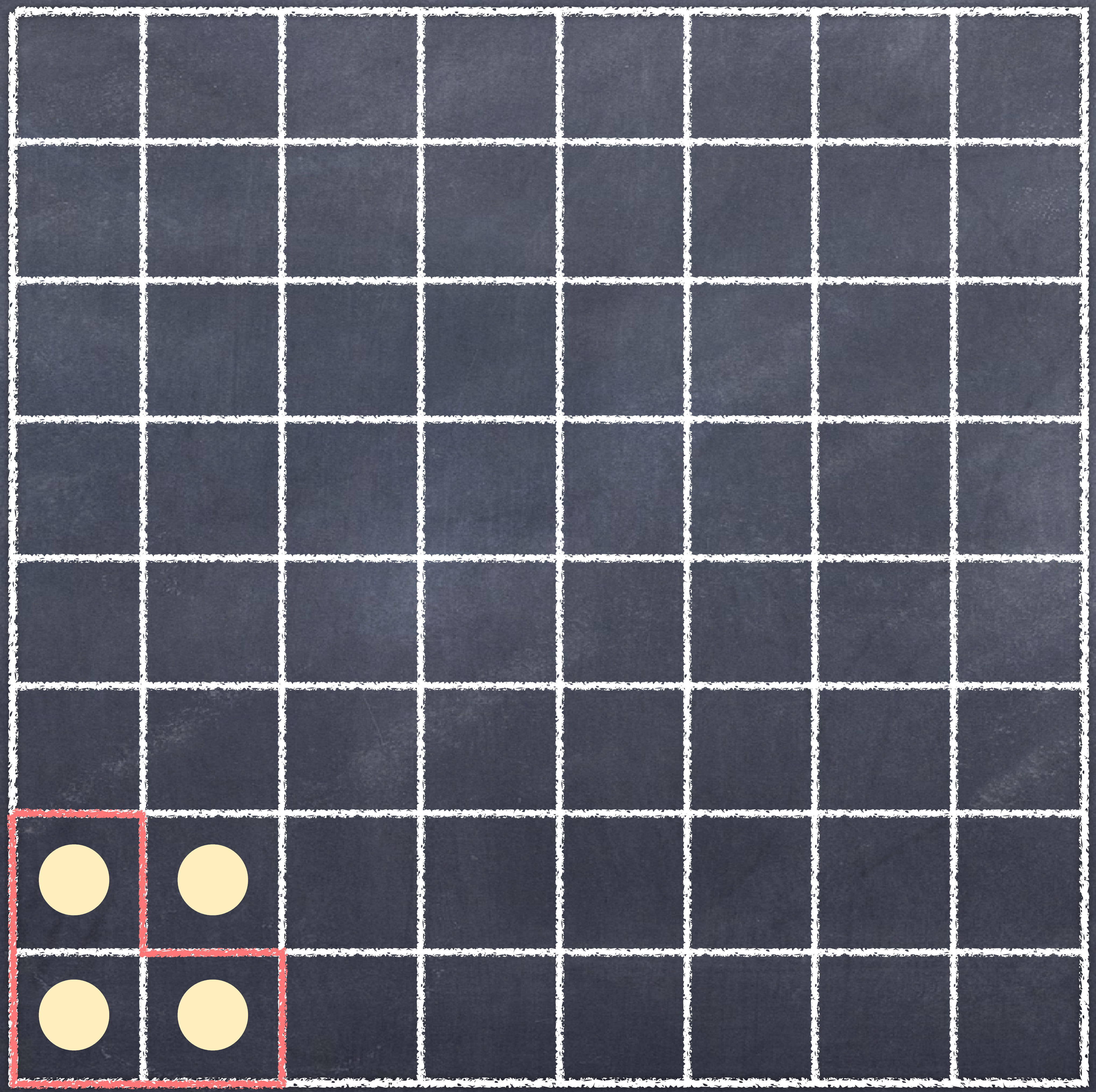
Clones into two aliens
at positions
(x+1,y)
&
(x,y+1)



One slot can accommodate
AT MOST one alien.

So the move is legal
only if the locations
(x+1,y)
&
(x,y+1)
are BOTH empty.

Freeing the aliens



Freeing the aliens

Try it yourself

Freeing the aliens

A configuration C is
a subset of locations on the board
that are occupied by aliens.

Freeing the aliens

Let C_0 denote the initial configuration, i.e:

$$C_0 = \{(0,0),(0,1),(1,0)\}$$

Let $\mathscr{R}$ denote the the set of ALL possible configurations that are reachable from C_0 by a finite sequence of legal moves.

Freeing the aliens

Will show:

There is no sequence of finite moves that makes the “prison” empty.

Freeing the aliens

Strategy

Associate a number
with every location on the board

$$f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}.$$

Freeing the aliens

Strategy

The weight of a configuration C with respect to f is defined as:

$$w_f(C) := \sum_{x \in C} f(x).$$

Freeing the aliens

Strategy

Choose f such that the following holds:

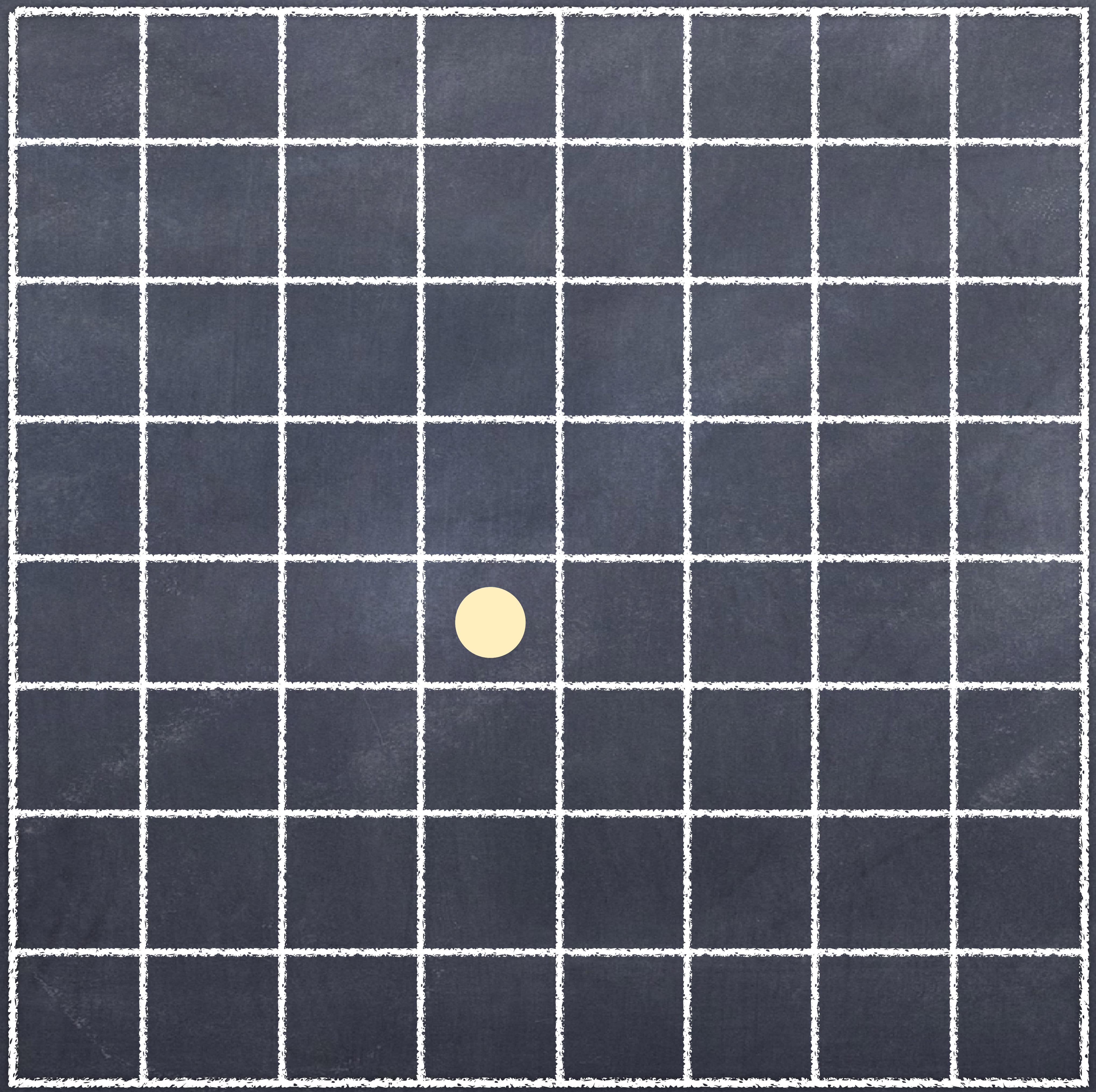
$$w_f(C_0) = w_f(C) \qquad \forall C \in \mathscr{R}$$

Freeing the aliens

In one move…

Alien at position (x,y)

Clones into two aliens
at positions
$(x+1,y)$
&
$(x,y+1)$



One slot can accommodate
AT MOST one alien.

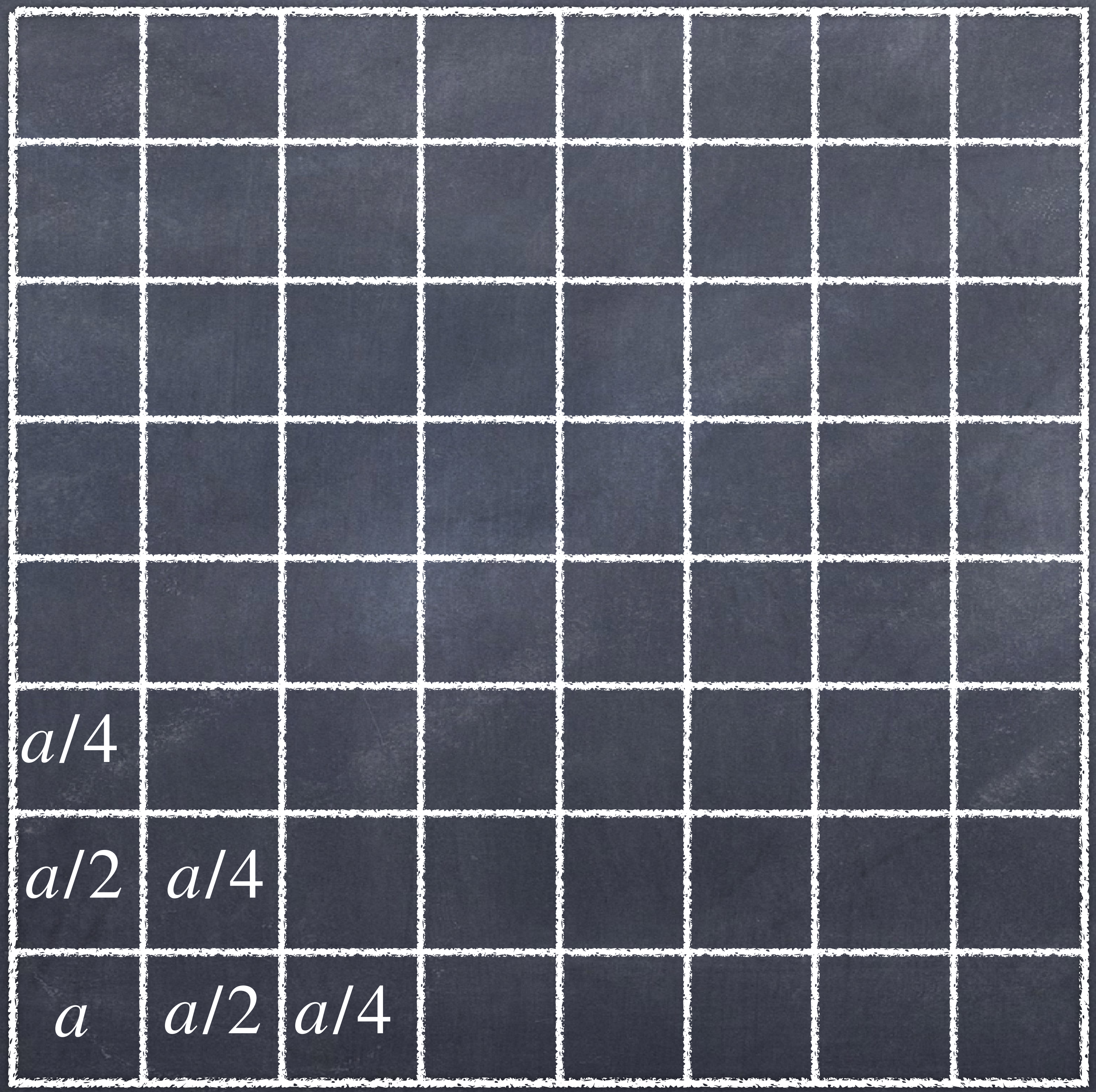
So the move is legal
only if the locations
$(x+1,y)$
&
$(x,y+1)$
are BOTH empty.

Freeing the aliens

In one move...

Alien at position (x,y)

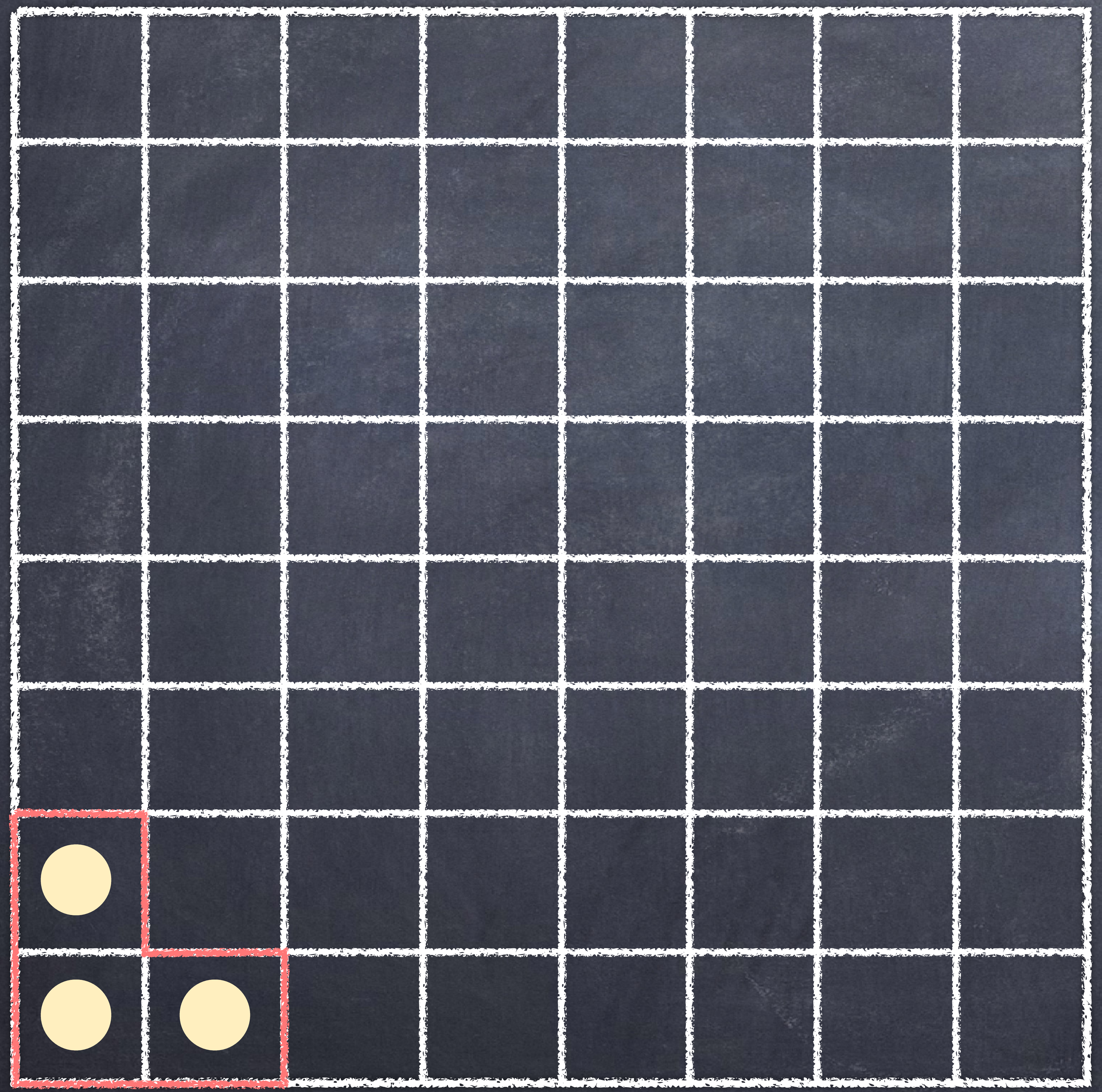
Clones into two aliens at positions (x+1,y) & (x,y+1)



One slot can accommodate AT MOST one alien.

So the move is legal only if the locations (x+1,y) & (x,y+1) are BOTH empty.

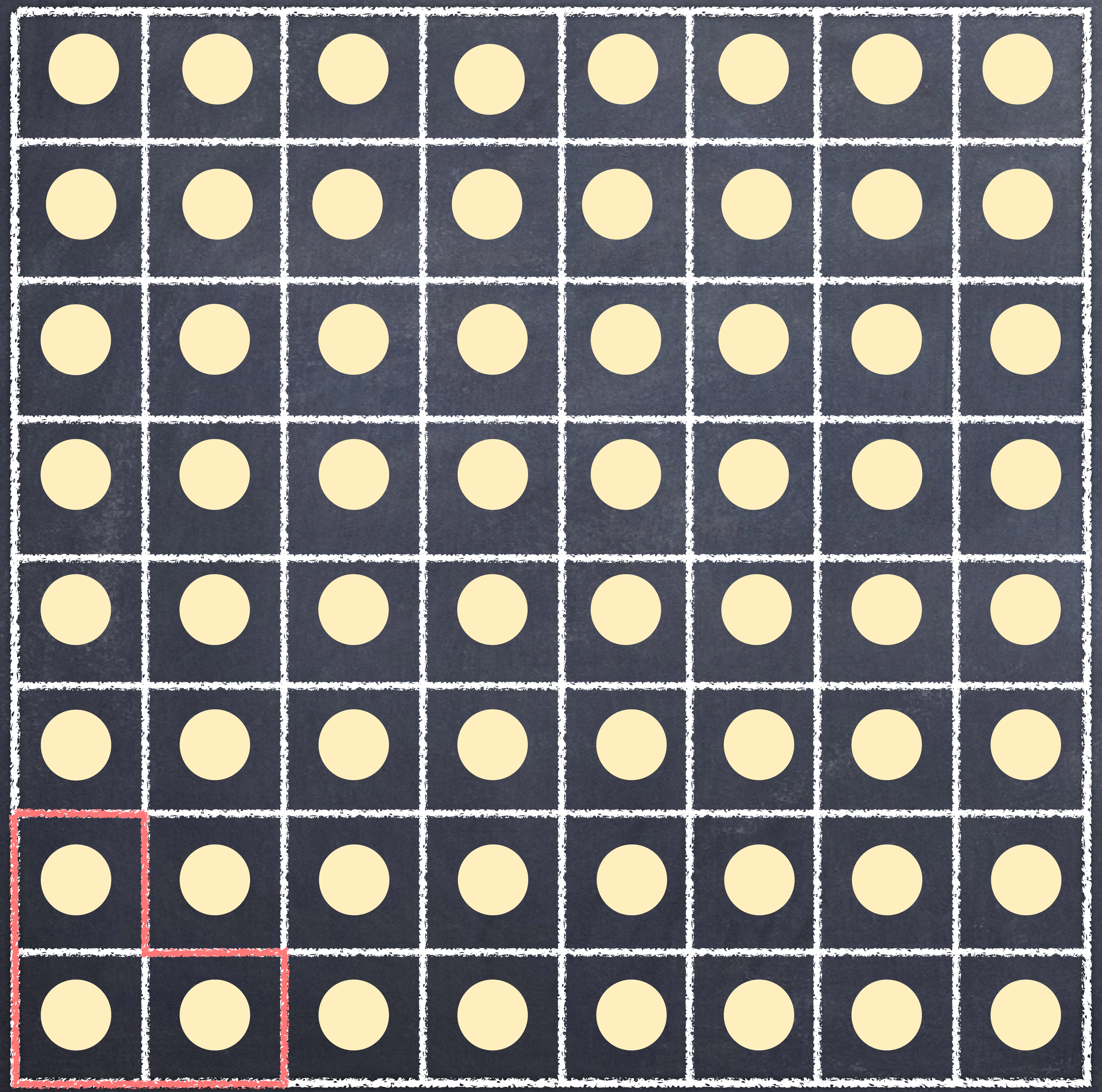
Freeing the aliens



Now note that:

$$w_f(C_0) = a + a/2 + a/2 = 2a$$

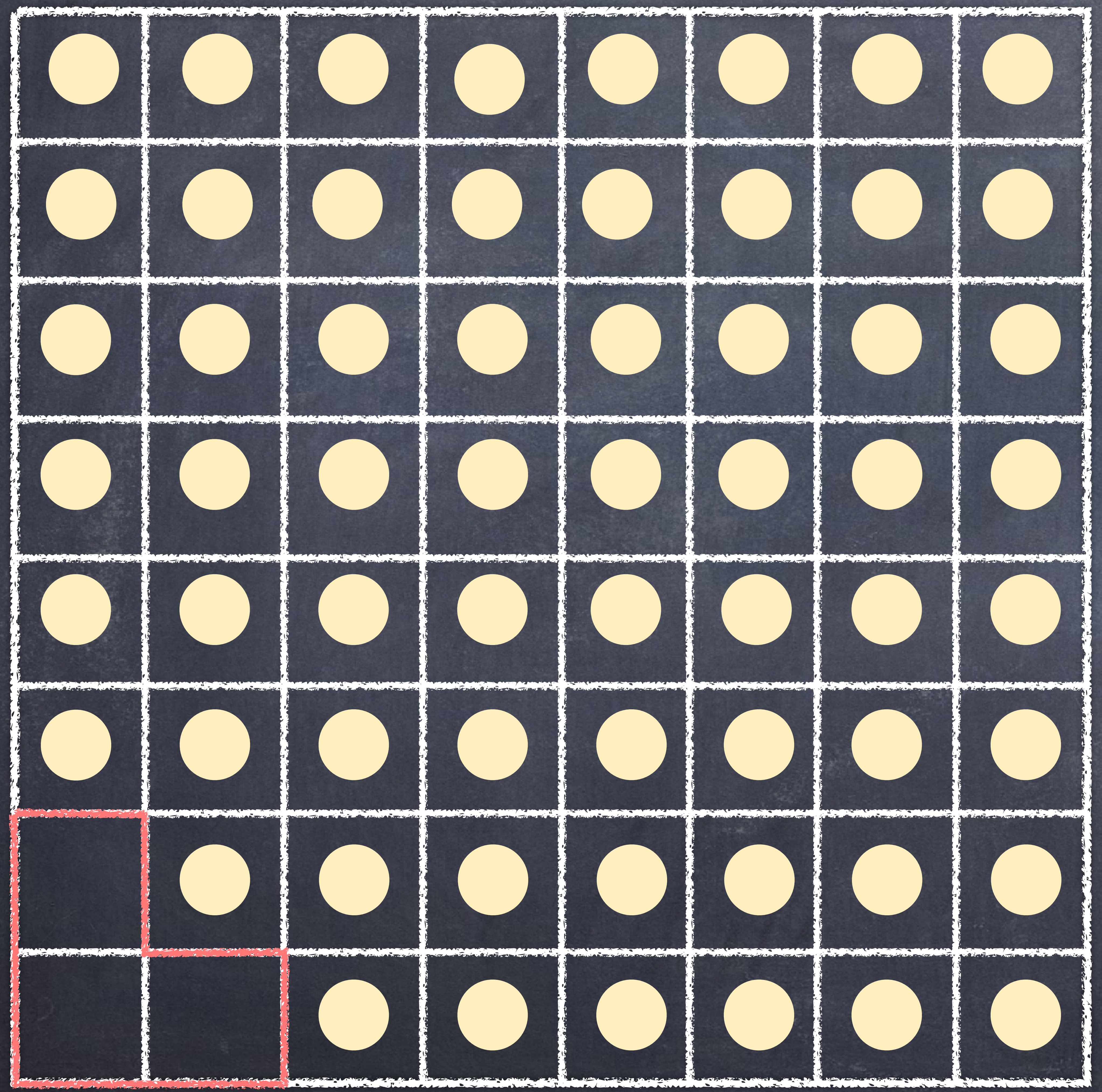
Freeing the aliens



It can be shown that:

$$w_f(C^\star) = 4a$$

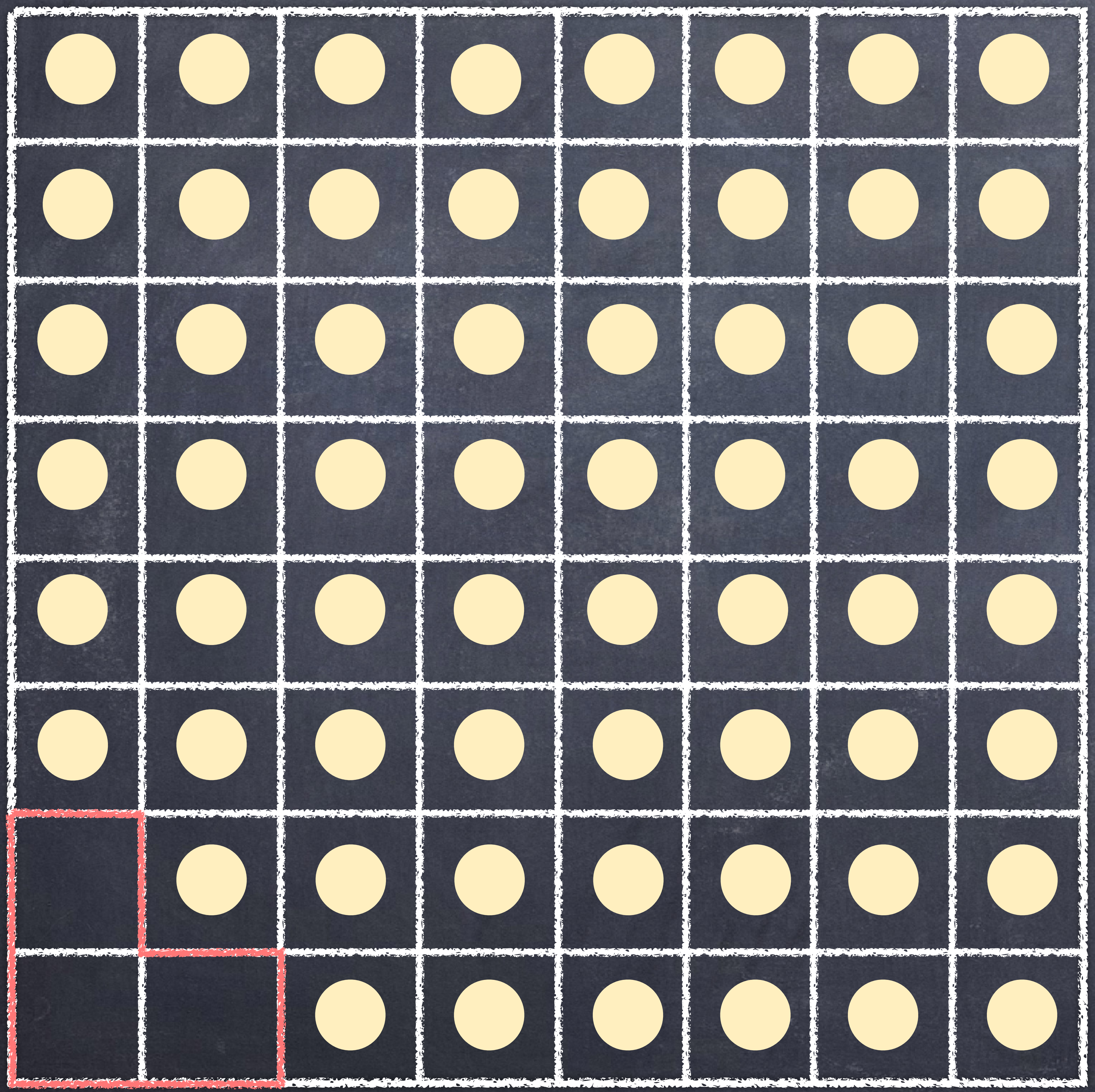
Freeing the aliens



Also, if
$C^{\dagger} := C^{\star} \setminus \{(0,0), (0,1), (1,0)\}$,
then:

$$w_f(C^{\dagger}) = 4a - 2a = 2a$$

Freeing the aliens

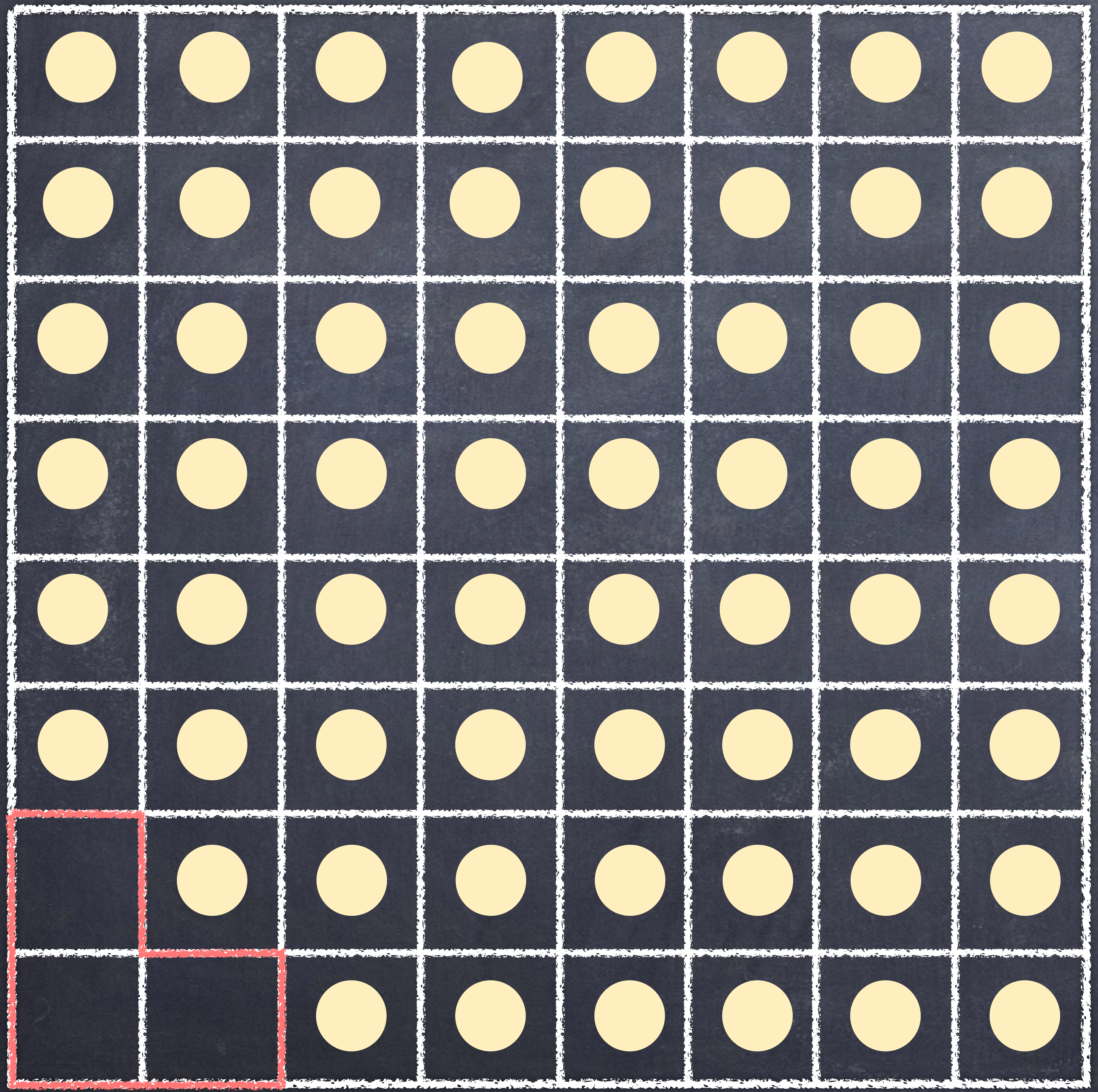


Suppose we arrive at a configuration D after a finite number of legal moves starting from C_0 where the aliens are freed from the prison.

Note that $D \subset C^\dagger$

So: $w_f(D) < w(C^\dagger) = 2a$

Freeing the aliens



Note that $D \subset C^{\dagger}$

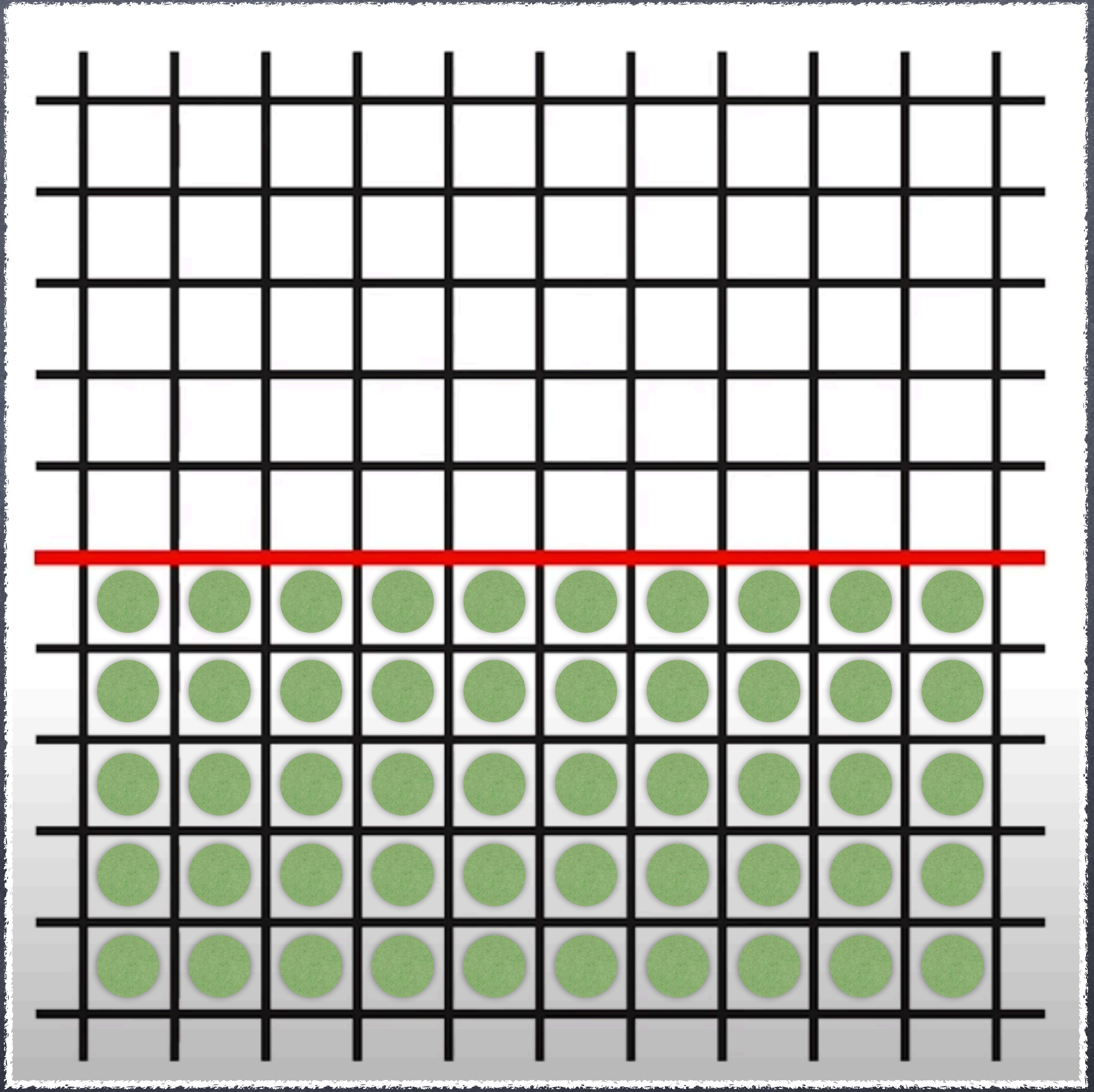
So: $w_f(D) < w(C^{\dagger}) = 2a$

But this contradicts the invariant (recall that $w(C_0) = 2a$ and the weight remains the same after any legal move.)

Conway's Checkers

Conway's Checkers

<u>Problem:</u> The bottom half of an Infinite checkerboard is populated with checkers, namely, at all (x, y) with $y \leq 0$. Using horizontal / vertical jumps, and removing the checker that is moved over, how high can you move a checker?



The main theorem

A checker from the lower half of the board

can't be moved to the fifth row

on the top half in finite number of legal moves.

Strategy for the proof: Find a monovariant!

We know what is an invariant. For instance, the parity of a permutation is an invariant.

But what is a monovariant?

A monovariant, unlike an invariant, is a numerical quantity that is either increasing or decreasing.

Our aim is to assign a quantity, call it the weight, to any configuration of the checkers on a board so that when we perform a legal move, the weight of the new configuration either increases, or remains the same.

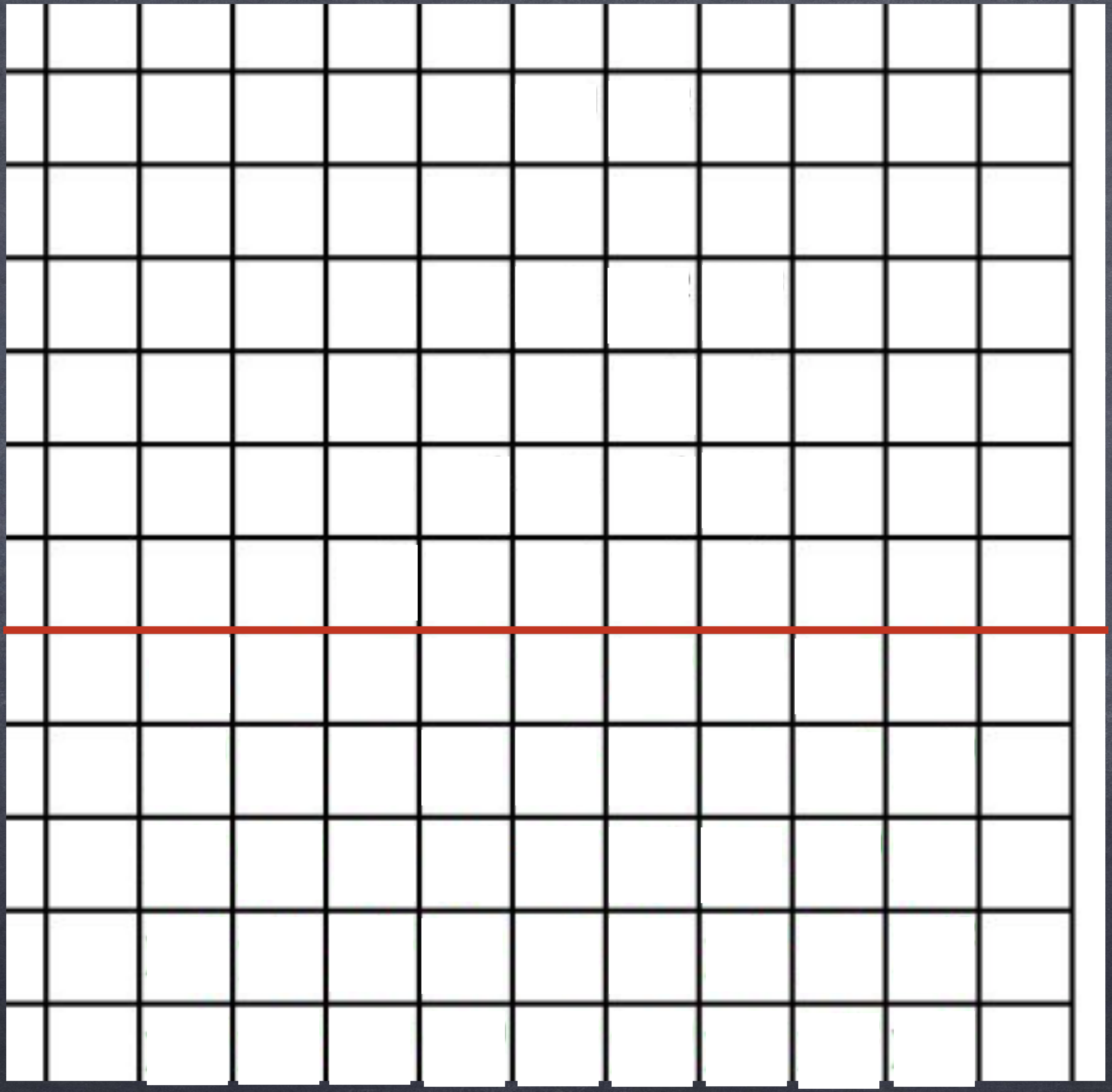
Guided by the example of freeing the aliens from the prison, we assign a number, call it the value, to each of the squares in the checker board. We define the weight to be the sum of all the values of the occupied squares.

Hope: After any legal move, the weight of the new configuration either remains unchanged or decrease.

Our goal is to put a checker in row 5 of the upper half of the board.

To prove that it is impossible, assume to the contrary that we have succeeded in moving one of the checkers to the fifth row. This position may be taken to be $(0,5)$ without loss of generality.

To assign a value to the square i units horizontally and j units vertically from $(0,5)$, fix x, $0 < x < 1,$ (to be determined later) and assign the value x^{i+j} to this square.



Two types of moves

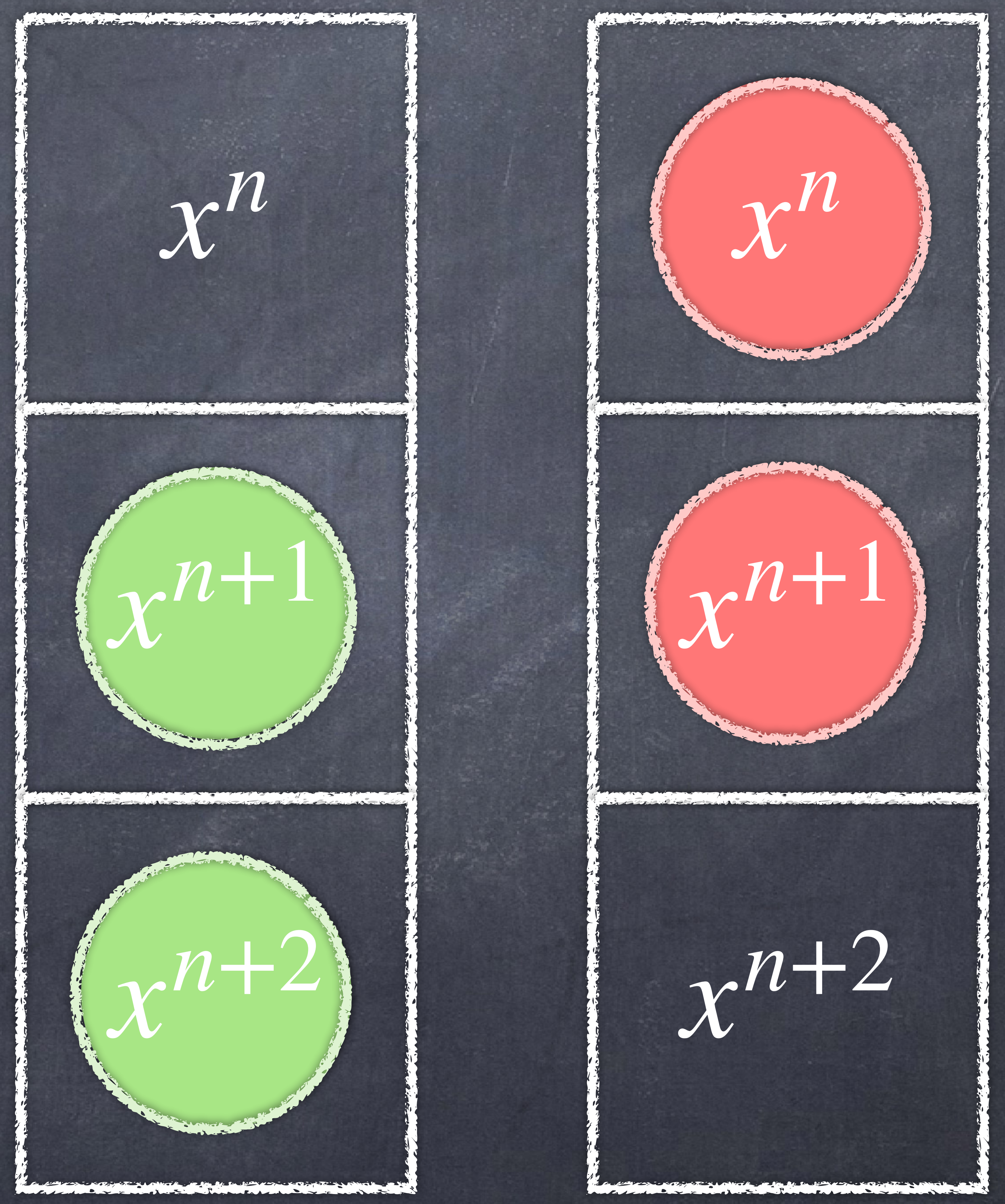
There are two types of moves:

Lose 2 checkers and add a one closer to $(0,5)$.

Lose 2 checkers and add one further from $(0,5)$.

The second type of move clearly decreases the weight of board.

For the first type, we need the weight of the board to remain unchanged.



Keeping the weight unchanged after a move of the second type amounts to choosing x such that

$$x^{n+2} + x^{n+1} = x^n, \text{ or equivalently, } x^2 + x = 1.$$

Thus $x = \dfrac{\sqrt{5} - 1}{2}$, this is the reciprocal of the golden ratio.

With this choice of x, the weight of the configuration is a monovariant by definition.

To arrive at a contradiction, let us calculate the weight of the initial configuration, assuming that each of the squares in the lower half plane is occupied.

Zeroth row: $\ldots, x^7, x^6, x^5, x^6, x^7, \ldots$, the sum is

$$x^5 + 2\sum_{k=6}^{\infty} x^k = x^5 + \frac{2x^6}{1-x} = \frac{(1+x)x^5}{1-x}.$$

Each row below the zeroth row is x times the previous row. So, the weight of the initial configuration is:

$$\frac{(1+x)x^5}{1-x}\sum_{n=0}^{\infty} x^n = \frac{(1+x)x^5}{(1-x)^2}$$

Magic

Weight of the initial configuration = $\dfrac{(1+x)x^5}{(1-x)^2}$.

When you substitute $x = \dfrac{\sqrt{5}-1}{2}$, then this becomes 1.

Weight of the target configuration is at least 1.

Magic

Since no move can increase the weight, the only way to reach the target configuration is by removing all the checkers from the initial configuration (since if even one is left, along with the checker at (0,5), the total weight will be more than 1, contradicting the monovariant property).

Zeckendorf's theorem

Fibonacci numbers:

$F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ldots,\quad F_{n+2} = F_{n+1} + F_n$

Theorem: Every natural number n can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

For the existence, apply the greedy algorithm: Let $n \in [F_k, F_{k+1})$ for some $k \in \mathbb{N}$. Thus $n \neq F_{k-1} + F_k$, otherwise by the recurrence formula, $n = F_{k+1}$, and we have assumed $n < F_{k+1}$ arriving at a contradiction. Repeat with the remainder $r = n - F_k \in [1, F_{k-1})$.

Zackendorf decomposition is minimal

Theorem: The Zckendorf is summand minimal: No decomposition as a sum of Fibonacci numbers $(1,2,3,5,\ldots)$ has fewer summands in it.

Proof: If $\sum_k a_k F_k$ (with a_k non-negative integers), define the weighted index attached to this decomposition $D$to be

Index(D) = $\sum_k a_k \sqrt{k}$.

To finish the proof, show that Index(D) is a monovariant, end in the Zeckendorf decomposition, number summands never increased.

For more on this theme and details of the proof, see https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Games_Monovariants_UMassOct2022.pdf

THANKS!