Chapter 3

Momentum

1. Dynamics of a System of Particles & Conservation of Momentum

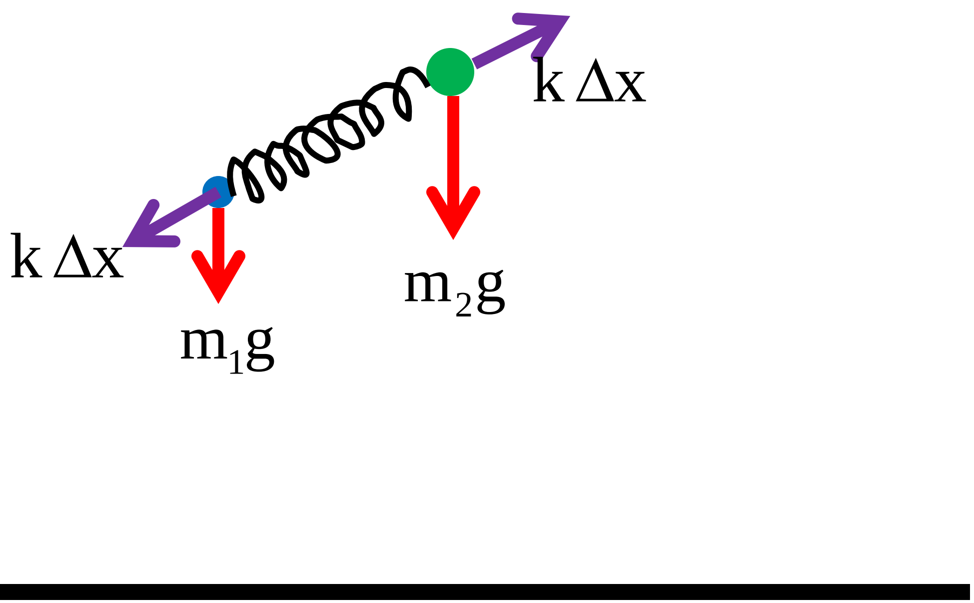
2. Concept of Centre of Mass

3. Motion of Systems with Variable Mass

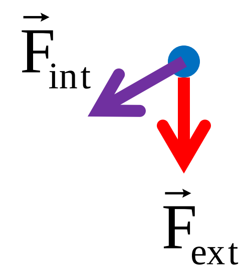
Dynamics of a System of Particles & Conservation of Momentum

The Two Particles System

Two masses connected by a mass less spring and falling in the gravitational field



Earth



Forces on First Mass

$$\frac{d\vec{p}_1}{dt} = \vec{F}_1^{ext} + \vec{F}_1^{int}$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_2^{ext} + \vec{F}_2^{int}$$

Adding the two

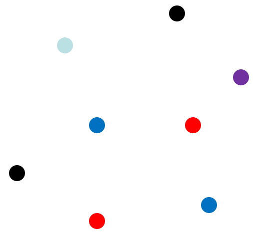
$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \left(\vec{F}_1^{int} + \vec{F}_2^{int}\right) + \left(\vec{F}_1^{ext} + \vec{F}_2^{ext}\right)$$

We have $\vec{F}_1^{int} + \vec{F}_2^{int} = \vec{0}$

$$\therefore \quad \frac{d\vec{p}_{tot}}{dt} = \vec{F}_{tot}^{ext}$$

Dynamics of a System of Indefinite Number of Particles

$$\vec{F}_j = \frac{d\vec{p}_j}{dt}$$



$$\vec{F}_j = \vec{F}_j^{\,int} + \vec{F}_j^{\,ext}$$

$$\sum_j \vec{F}_j^{\,int} + \sum_j \vec{F}_j^{\,ext} = \sum_j \frac{d\vec{p}_j}{dt}$$

$$\sum_j \vec{F}_j^{\,int} = \vec{0} \qquad \sum_j \vec{F}_j^{\,ext} = \vec{F}_{ext}$$

$$\sum_j \frac{d\vec{p}_j}{dt} = \vec{F}_{ext}$$

$$\text{Or,} \quad \frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

Where,

$$\vec{P} = \sum_j \vec{p}_j$$

is the total momentum of the system

Conservation of Momentum

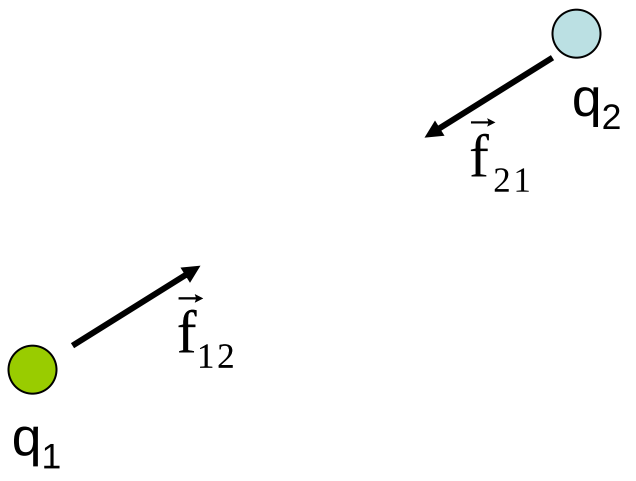
Since $$\frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

$$\vec{F}_{ext} = \vec{0} \Rightarrow \vec{P} = \text{Const.}$$

In the absence of any net external force, the total momentum of a system of particles is conserved, although individual momenta may change with time

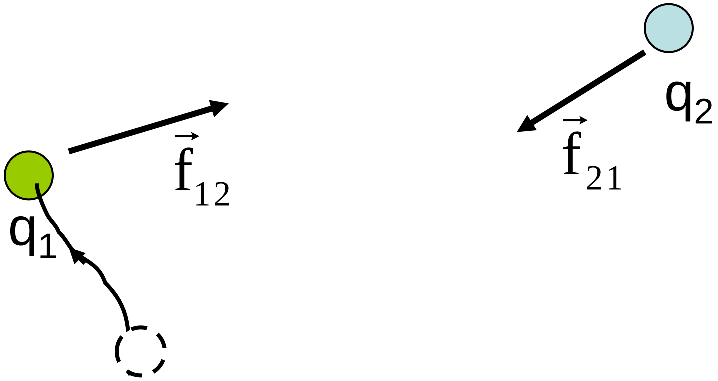
What if the internal forces do not add up to zero?

Electrostatics



$$\vec{f}_{12} = -\vec{f}_{21}$$

Electrodynamics



$$\vec{f}_{12} \neq -\vec{f}_{21}$$

In electrodynamics, momentum may not be conserved!

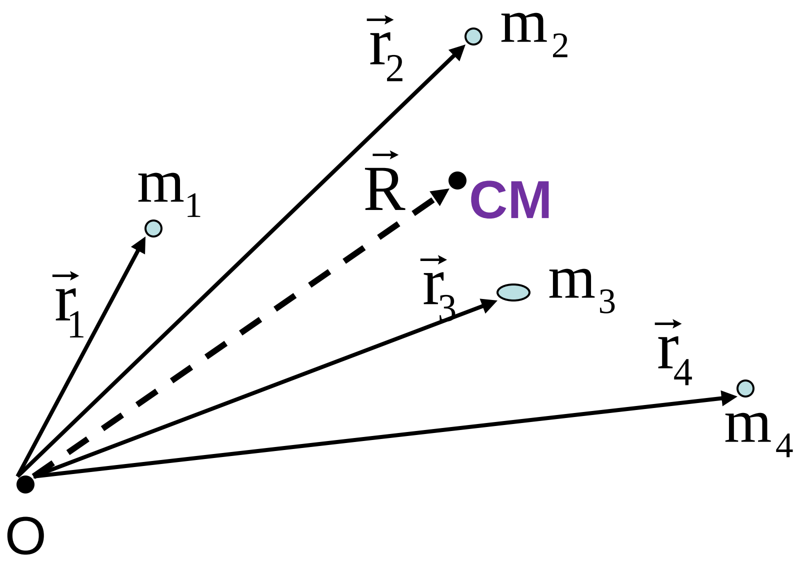
Centre of Mass

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 \ldots\ldots + m_N\vec{r}_N}{m_1 + m_2 \ldots\ldots + m_N}$$

$$X = \frac{m_1x_1 + m_2x_2 \ldots\ldots + m_Nx_N}{m_1 + m_2 \ldots\ldots + m_N}$$

$$Y = \frac{m_1y_1 + m_2y_2 \ldots\ldots + m_Ny_N}{m_1 + m_2 \ldots\ldots + m_N}$$

$$Z = \frac{m_1z_1 + m_2z_2 \ldots\ldots + m_Nz_N}{m_1 + m_2 \ldots\ldots + m_N}$$



Motion of CM

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$M\dot{\vec{R}} = \sum_i m_i \dot{\vec{r}}_i = \sum_i \vec{p}_i = \vec{P}$$

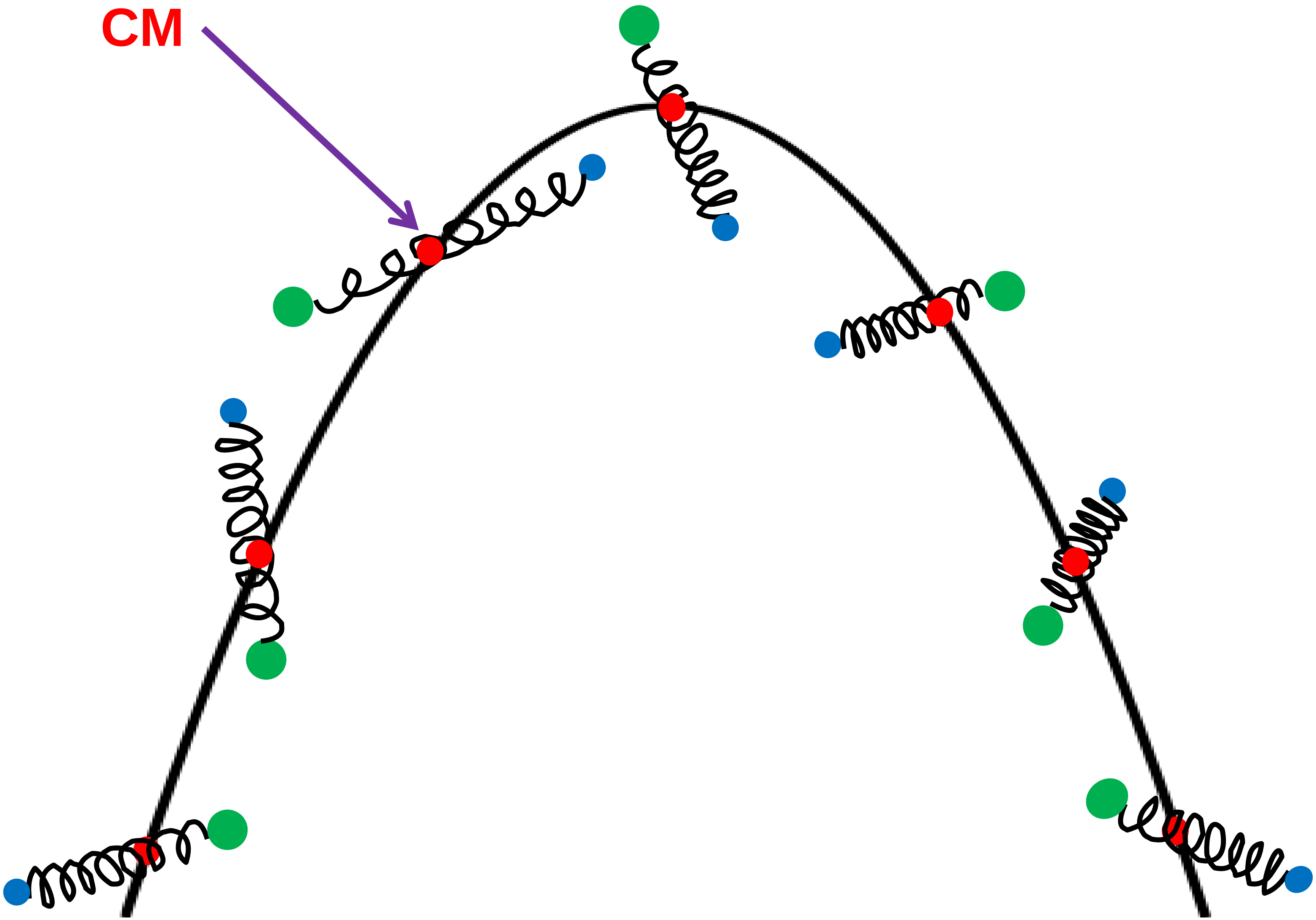
$$M\frac{d^2\vec{R}}{dt^2} = \frac{d\vec{P}}{dt} = \vec{F}_{ext}$$ **Or** $$M\vec{a}_{CM} = \vec{F}_{ext}$$

The centre of mass of a system of particles moves as though it is particle, carrying the total mass of the system, and is acted upon by the net external force on the system.

$$\vec{F}_{ext} = \vec{0} \Rightarrow \vec{V} = \frac{d\vec{R}}{dt} = \text{Const.}$$

$$\therefore \vec{R} = \vec{R}_0 + \vec{V}\,t$$

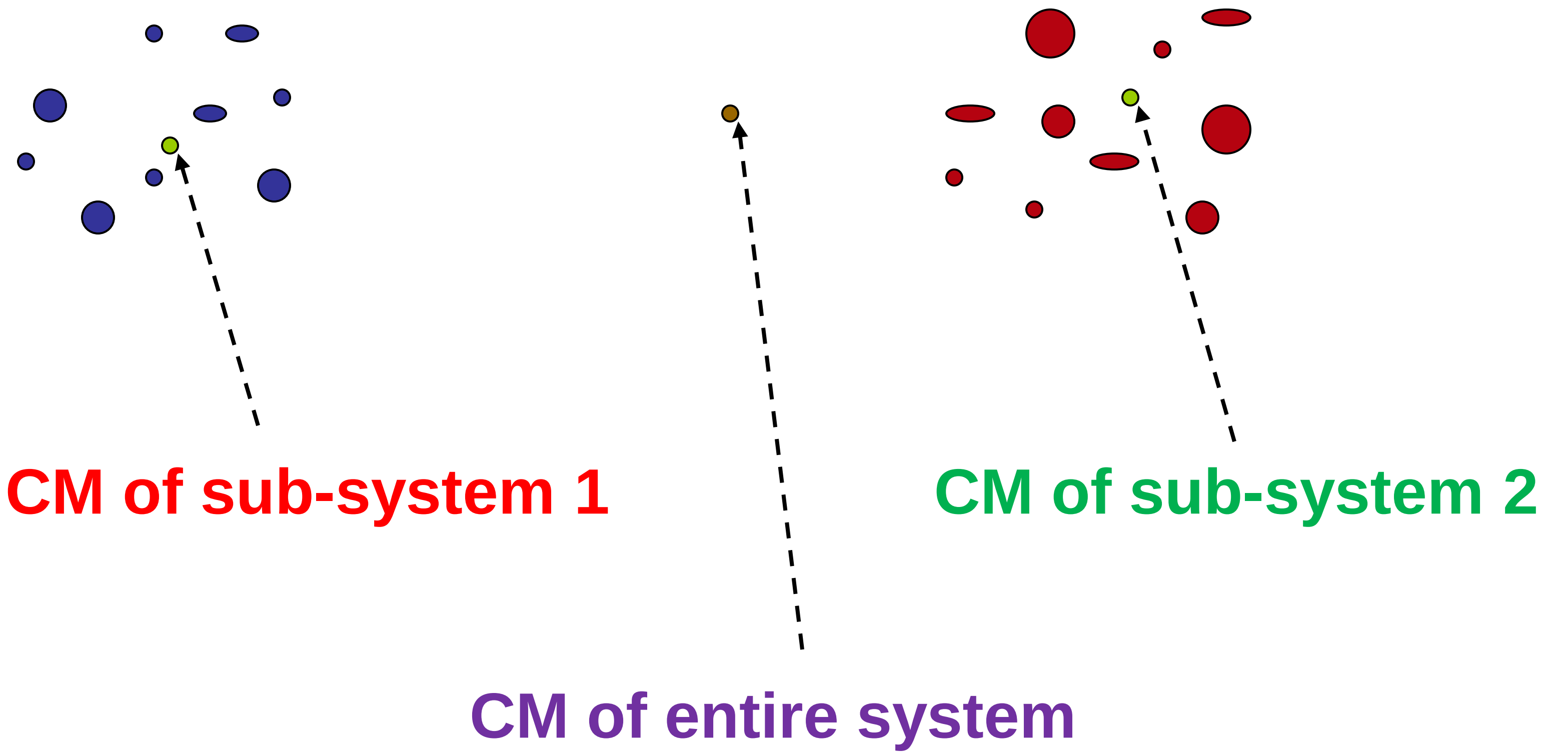
$\vec{R}_0$ **is the position of CM at t = 0**

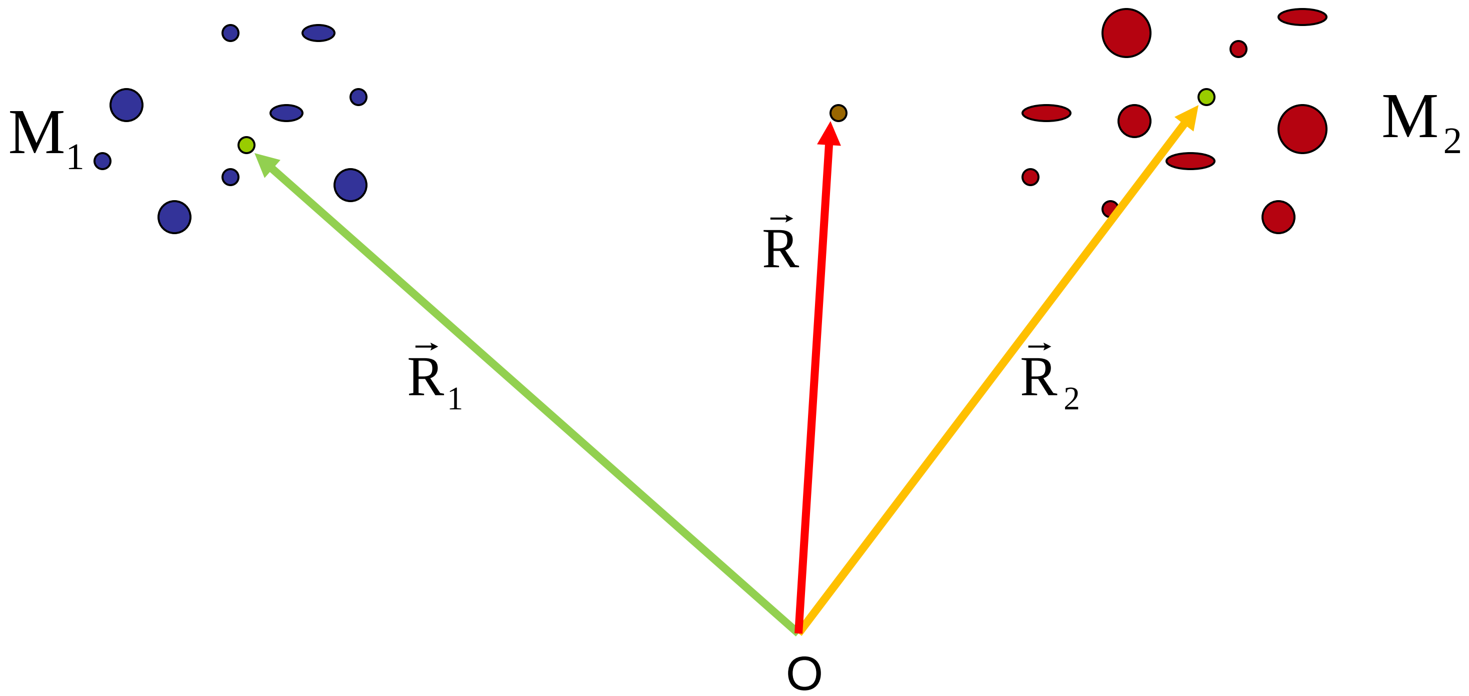


Spring-mass system tossed into the air

Prob. 3.3

Suppose that a system consists of several bodies, and the position of the CM of each body is known. Prove that the CM of the combined system can be found by treating each body as a particle concentrated at its own CM.


CM of sub-system 1
CM of sub-system 2
CM of entire system



To prove : $$\vec{R} = \frac{M_1 \vec{R}_1 + M_2 \vec{R}_2}{M_1 + M_2}$$

M_1 : Mass of Sub-system 1

M_2 : Mass of Sub-system 2

$$\vec{R}_1 = \frac{\sum_{i=1}^{M} m_i \vec{R}_i}{M_1} \quad ; \quad \vec{R}_2 = \frac{\sum_{i=M+1}^{M+N} m_i \vec{R}_i}{M_2}$$

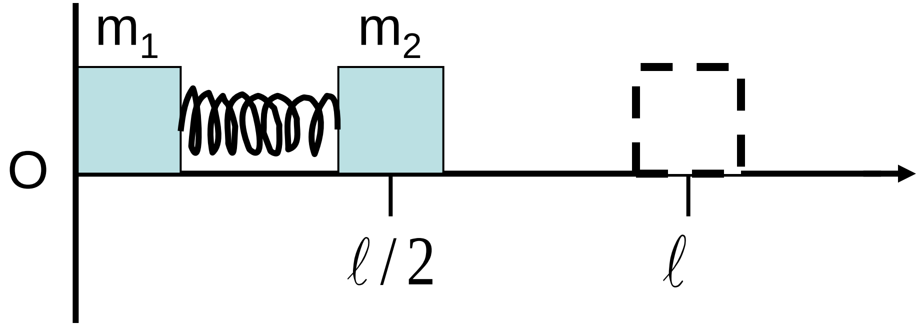
$$\therefore \quad \frac{M_1 \vec{R}_1 + M_2 \vec{R}_2}{M_1 + M_2} = \frac{\sum_{i=1}^{M} m_i \vec{R}_i + \sum_{i=M+1}^{M+N} m_1 \vec{R}_i}{M_1 + M_2}$$

$$= \frac{\sum_{i=1}^{M+N} m_i \vec{R}_i}{M_{tot}} = \vec{R}$$

Prob. 3.7

Two masses, m_1 & m_2 are connected by a spring of spring constant k and unstretched length L. The entire system is pushed against a wall so that the spring is compressed to length L/2. Mass m_2 is then released at t = 0.

Find the motion of the CM as function of time.



The mass m_1 will not leave the wall until m_2 reaches position ℓ

Motion of m_2 from $x=\ell/2$ to $x=\ell$ is SHM of amplitude $\ell/2$ and frequency $\omega=\sqrt{k/m_2}$

1. For time $0 \le t \le \dfrac{\pi}{2\omega}$

$$x_1(t) = 0 \;\; ; \;\; x_2(t) = \ell - \frac{\ell}{2}\cos\omega t$$

$$\Rightarrow X(t) = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\ell}{2}\left[1 - \frac{1}{2}\cos\omega t\right]$$

(Assuming equal masses)

2. For time $t > \dfrac{\pi}{2\omega}$

CM moves uniformly forward

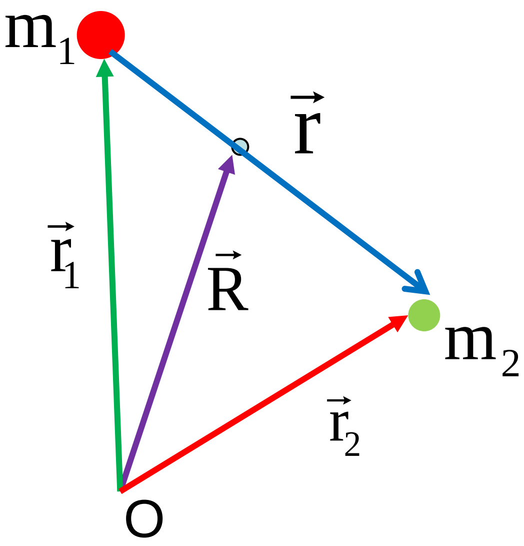
$$\therefore \quad X(t) = \ell/2 + V(\pi/2\omega)\,t = \ell/2 + \frac{\ell\omega}{4}\,t$$

CM Coordinates and Reduced Mass

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_1$$

$$m_2 \ddot{\vec{r}}_2 = \vec{F}_2$$

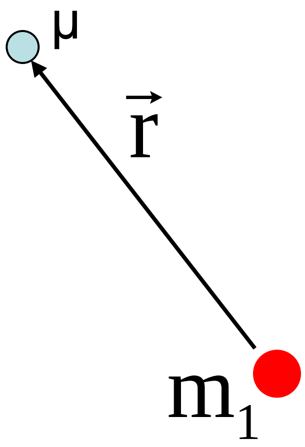
Define : $\vec{r} = \vec{r}_2 - \vec{r}_1$



$$\Rightarrow \ddot{\vec{r}} = \frac{\vec{F}_2}{m_2} - \frac{\vec{F}_1}{m_1}$$

$$= \left(1/m_1 + 1/m_2\right)\vec{F}_2 = \frac{\vec{F}_2}{\mu}$$

$$\text{or,}\quad \mu\ddot{\vec{r}} = \vec{F}_2$$



Where,

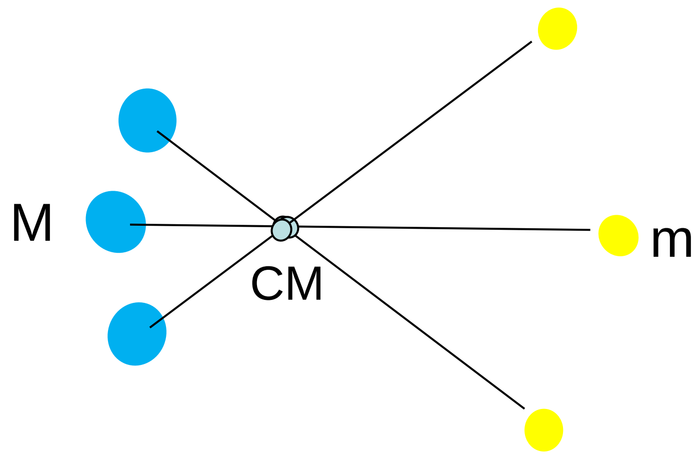
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass of the system

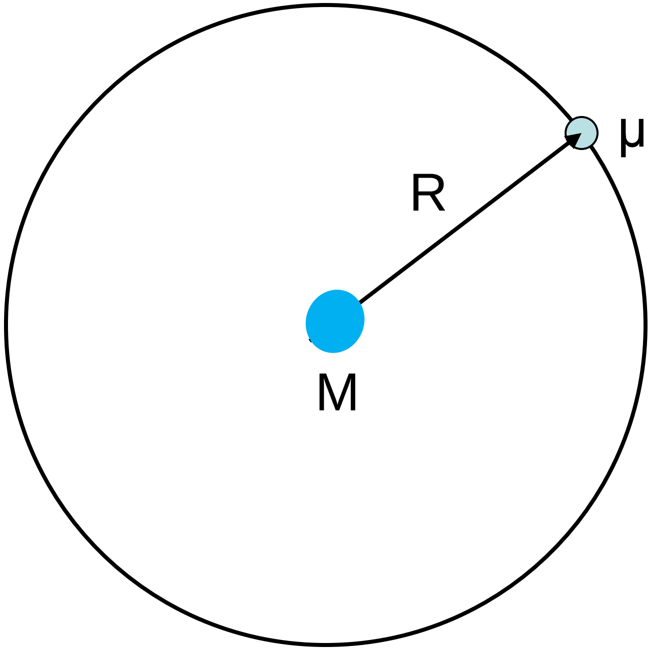
$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r} \quad ; \quad \vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r}$$

Prob. 2.4

Two masses m & M respectively, undergo uniform circular motion about each other at a separation of R, under the influence of their gravitational attraction. Find the time period of rotation.



Motion observed by an Outside Inertial Observer



Motion of ‘m’ seen from ‘M’

$$\mu R \omega^2 = \frac{GMm}{R^2} \quad \Rightarrow \quad \omega = \sqrt{\frac{GMm}{\mu R^3}} \quad = \sqrt{\frac{G(M+m)}{R^3}}$$

$$\Rightarrow T = 2\pi \frac{R^{3/2}}{\sqrt{G(M+m)}}$$

For the earth-moon system, $\frac{m}{M} \approx \frac{1}{80}$

Thus, the correction due to the motion of the earth is a reduction in time period of the moon by $\approx 0.62\%$

Impulse and Momentum

Impulse of a force over the time interval

t_1 to t_2

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$$

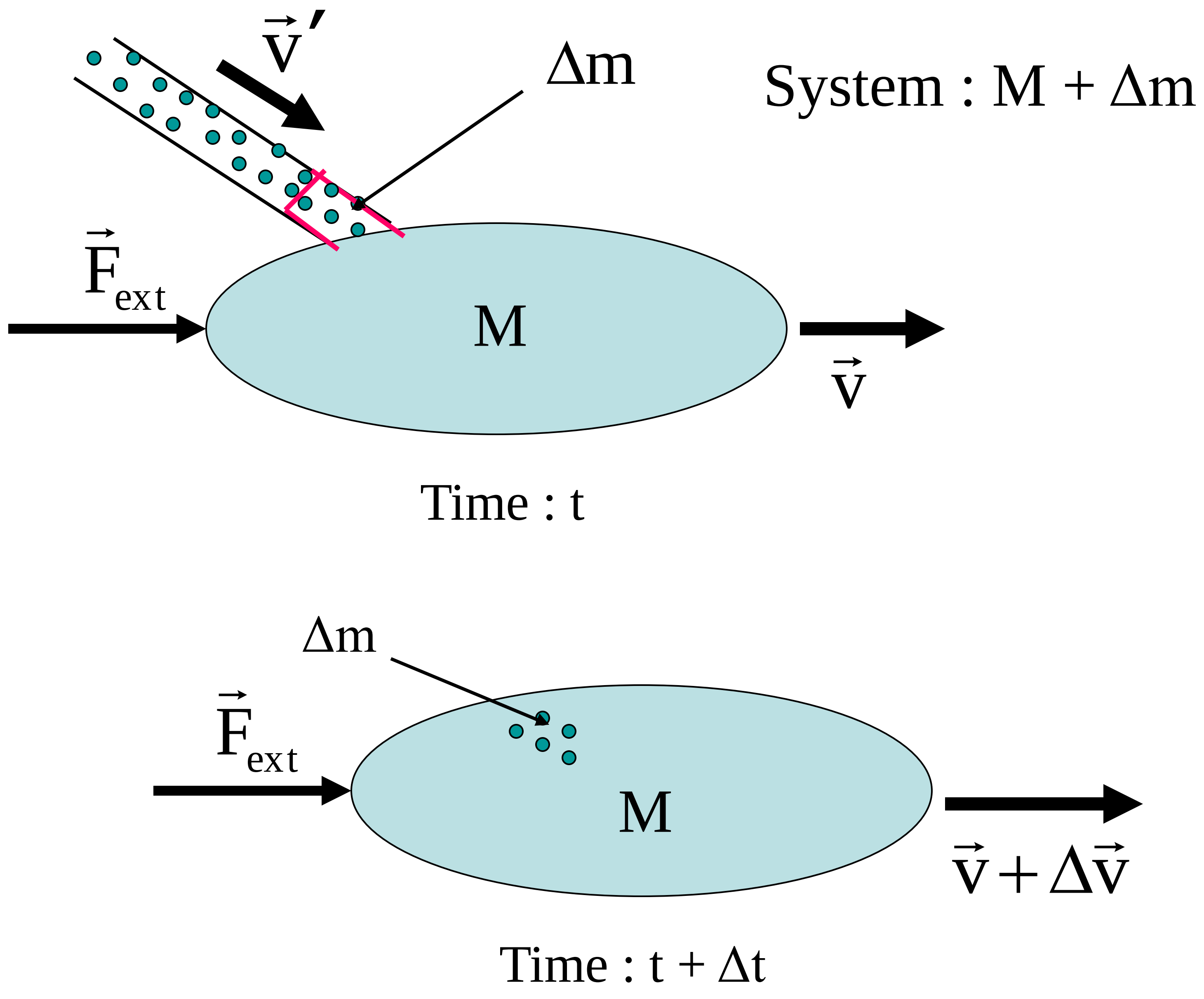
Putting $\vec{F} = \dfrac{d\vec{p}}{dt}$

$$\vec{I} = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt = \int_{t_1}^{t_2} d\vec{p} = \vec{p}(t_2) - \vec{p}(t_1) = \Delta\vec{p}$$

Systems With Variable Mass

1. Mass Continually Added to System :
Loading of a moving wagon, Conveyer belts

2. Mass Continually Leaving The System :
Rocket, Leaking wagon



$\vec{v}'$
Δm
System : M + Δm
$\vec{F}_{ext}$
M
$\vec{v}$
Time : t
Δm
$\vec{F}_{ext}$
M
$\vec{v} + \Delta\vec{v}$
Time : t + Δt

Change in the momentum over time Δt :

$$\Delta\vec{P} = (M+\Delta m)(\vec{v}+\Delta\vec{v}) - (M\vec{v} + \Delta m\,\vec{v}')$$

$$= \; M\Delta\vec{v} + \Delta m(\vec{v} - \vec{v}') + \Delta m \cdot \Delta\vec{v}$$

Equating this to the impulse of the external force on the system, $\vec{F}_{ext}\Delta t$, and dividing out by Δt

$$\vec{F}_{ext} = M\frac{\Delta\vec{v}}{\Delta t} + \frac{\Delta m}{\Delta t}(\vec{v} - \vec{v}') + \Delta m \cdot \frac{\Delta\vec{v}}{\Delta t}$$

In the limit as $\Delta t \to 0$, the last term drops out, and

$$M\vec{a} = \vec{F}_{ext} + \frac{dm}{dt}\,\vec{v}_{rel}$$

Where, $\vec{v}_{rel} = \vec{v}' - \vec{v}$, is the relative velocity of the mass entering/leaving the system w.r.t. the main mass

Prob. 3.20

A rocket ascends from rest in a uniform gravitational field by ejecting exhaust.

Speed of exhaust : u

Rate of exhaust : $dm/dt = \gamma m \quad (\gamma < 0)$
(m is instantaneous mass of the rocket)

Retarding force : bmv (v is instantaneous speed of rocket)

Find : velocity of rocket as function of time

Equation of motion

$$m\frac{dv}{dt} = -mg - bvm - \gamma um$$

$$\frac{dv}{dt} = -(bv + g + \gamma u)$$

$$\int_0^v \frac{dv'}{bv' + g + \gamma u} = -\int_0^t dt'$$

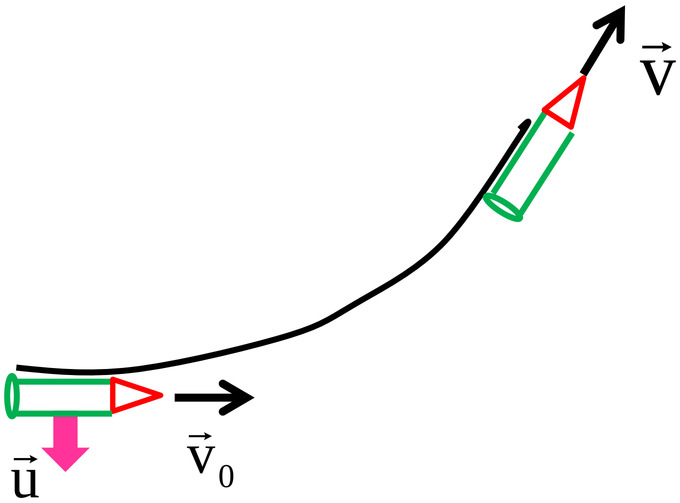
$$\frac{1}{b}\ln\left[\frac{bv + g + \gamma u}{g + \gamma u}\right] = -t$$

$$v(t) = \frac{g + \gamma u}{b}\left[e^{-bt} - 1\right]$$

$$= \frac{|\gamma|u - g}{b}\left[1 - e^{-bt}\right]$$

Example*

Change of Course by a Spaceship :



Rate of Exhaust = b $\Rightarrow$ $M(t) = M_0 - bt$

$$\vec{u} \cdot \vec{v} = 0 \qquad |\vec{u}| = u_0$$

Equation of Motion

$$M\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{v}_{rel} = -\,b\vec{u}$$

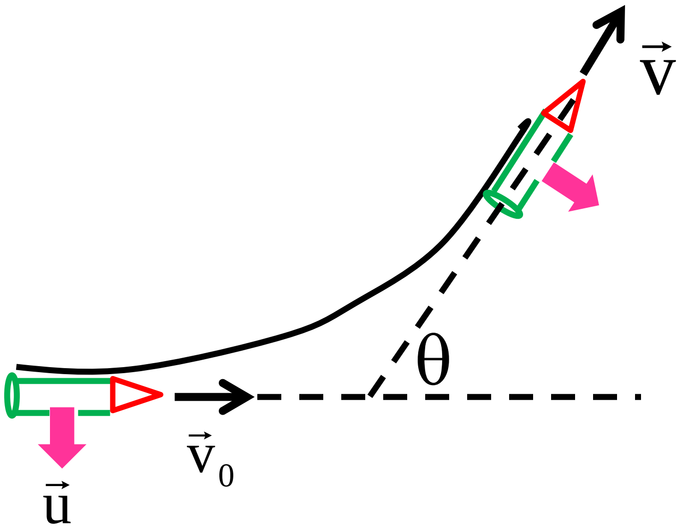
Taking dot product with $\vec{v}$

$$M\vec{v}\cdot\frac{d\vec{v}}{dt} = -\,b\vec{v}\cdot\vec{u} = 0$$

$$\Rightarrow \quad |\vec{v}| = \text{Const.} = v_0$$

Taking dot product with $\vec{v}_0$

$$\Rightarrow \quad M\vec{v}_0\cdot\frac{d\vec{v}}{dt} = -\,b\vec{v}_0\cdot\vec{u}$$



$$\vec{v}_0 \cdot \vec{v} = v_0^2 \cos\theta \qquad \vec{v}_0 \cdot \vec{u} = v_0 u_0 \sin\theta$$

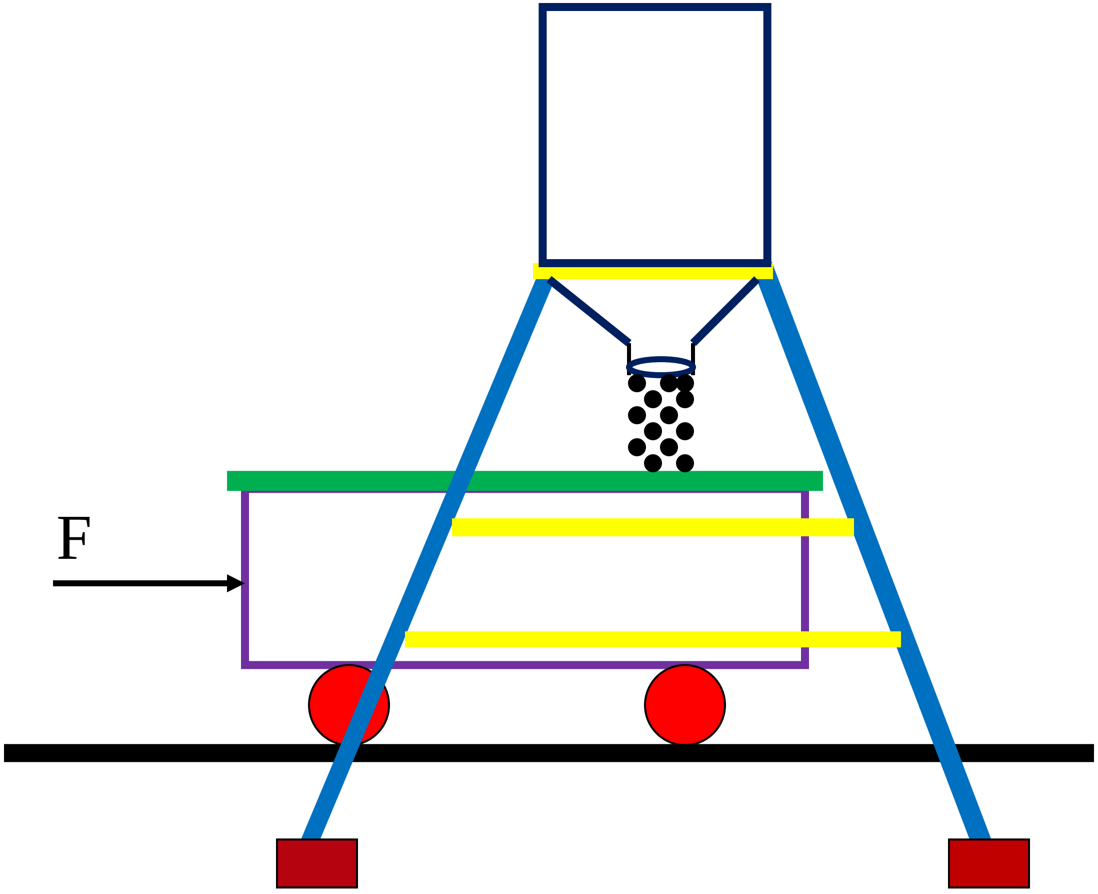
$$\therefore \quad M v_0^2 \frac{d}{dt}(\cos\theta) = -\, b\, v_0 u_0 \sin\theta$$

$$\frac{d\theta}{dt} = \frac{b u_0}{v_0} \frac{1}{M_0 - bt} \qquad \Rightarrow \quad \theta(t) = \frac{u_0}{v_0} \ell n \left[\frac{M_0}{M_0 - bt} \right]$$

Prob. 3.10

An empty freight car of mass M starts from rest under an applied force F. At the same time sand begins to run into the car at a steady rate b from a hopper at rest along the track.

Find the speed of the car after a mass of sand m has been transferred.


F

Here,

$$\text{i)}\ \frac{dm}{dt} = b \qquad \text{ii)}\ v_{rel} = -v \qquad \text{iii)}\ M = M_0 + bt$$

$\therefore$ **Equation of Motion**

$$M\frac{dv}{dt} = F - bv$$

$$\therefore \quad \int_0^v \frac{dv'}{F - bv'} = \int_o^{m/b} \frac{dt'}{M_0 + bt'}$$

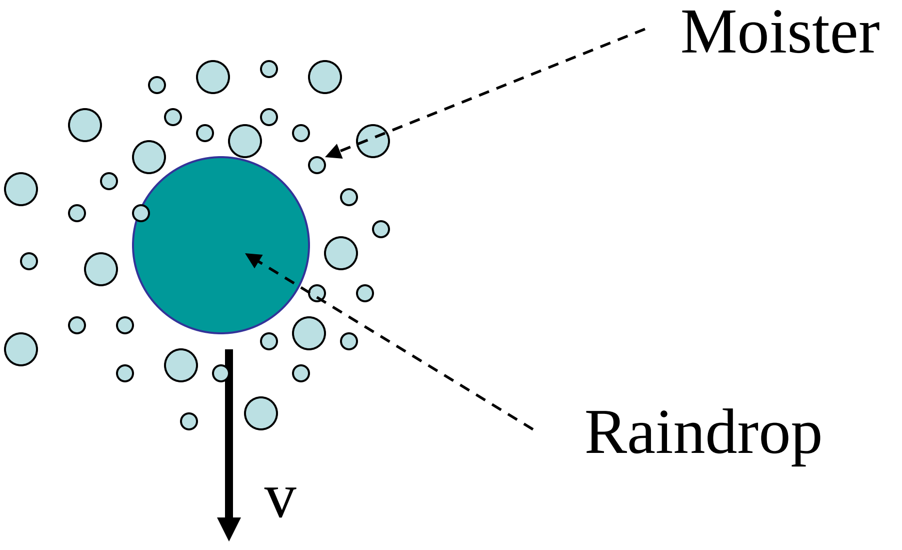
$$-\frac{1}{b}\ln\left[\frac{F - bv}{F}\right] = \frac{1}{b}\ln\left[\frac{M_0 + m}{M_0}\right]$$

Prob. 3.18

Raindrop of initial mass M_0 falls from rest under the influence of gravity. The drop gains mass at a rate proportional to the product of its instantaneous mass and instantaneous velocity :

$$\frac{dM}{dt} = k\,M\,v$$

Show that the speed of the drop eventually becomes effectively constant and give an expression for the terminal speed. Neglect air resistance



Equation of Motion

$$M\frac{dv}{dt} = Mg - kMv^2 \qquad (\because \; v_{rel} = -v)$$

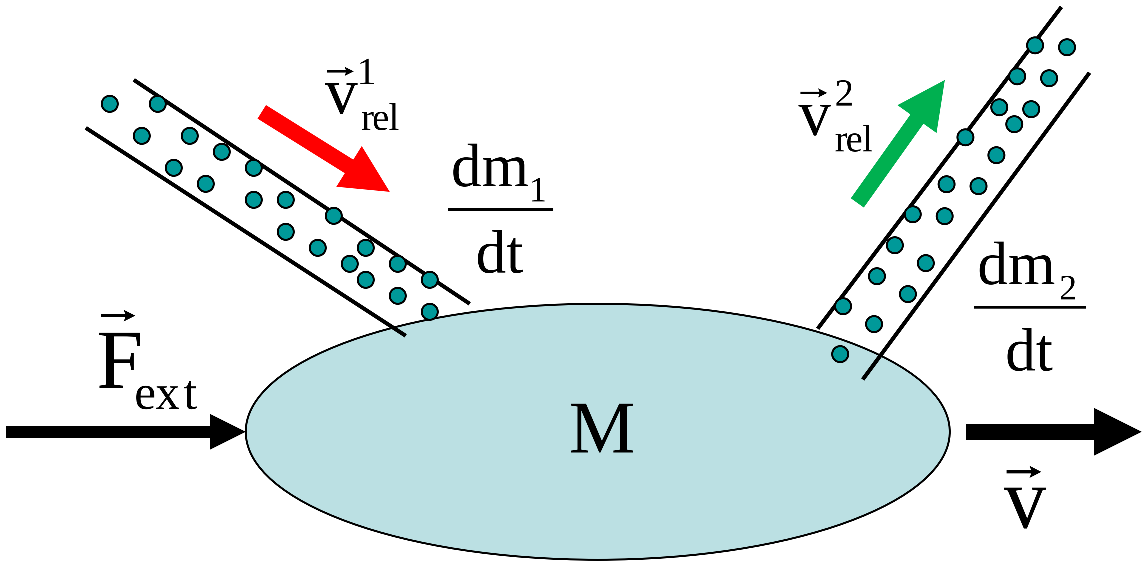
$$\Rightarrow \quad \int_o^v \frac{dv'}{g - kv'^2} = \int_0^t dt'$$

$$\int_0^v \frac{dv'}{\sqrt{g} + \sqrt{k}\, v'} + \int_0^v \frac{dv'}{\sqrt{g} - \sqrt{k}\, v'} = 2\sqrt{g}\, t$$

Integrating and solving for v,

$$v = \sqrt{\frac{g}{k}} \left[\frac{e^{2\sqrt{gk}\,t} - 1}{e^{2\sqrt{gk}\,t} + 1} \right]$$

$$\Rightarrow \quad v_{term} = \sqrt{\frac{g}{k}}$$



Equation of Motion

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \left(\frac{dm_1}{dt}\right)\vec{v}^{\,1}_{rel} + \left(\frac{dm_2}{dt}\right)\vec{v}^{\,2}_{rel}$$

Prob. 3.13

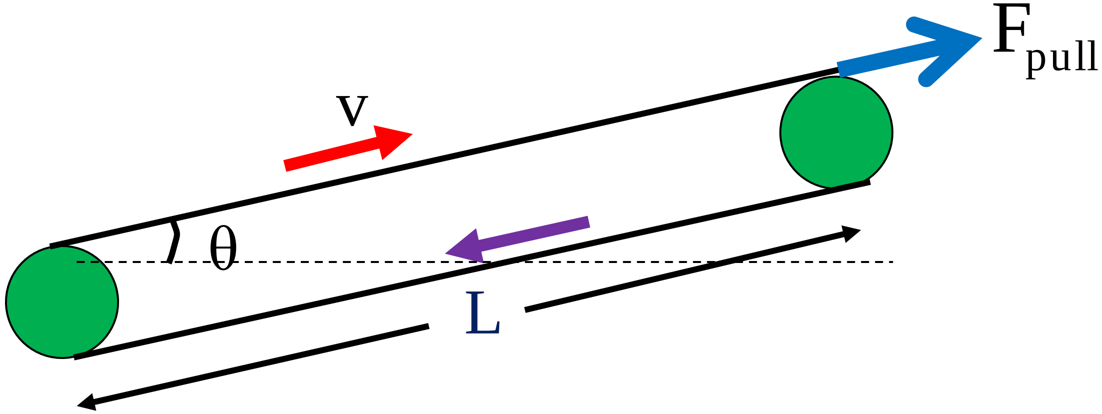
A ski tow consists of a long belt of rope around two pulleys, one at the bottom of a slope and the other at the top.

A motor drives the pulleys to move the belt at constant speed of 1.5 m/s.

Length of belt : 100 m

A skier of mass 70 kg takes the tow every 5 s.

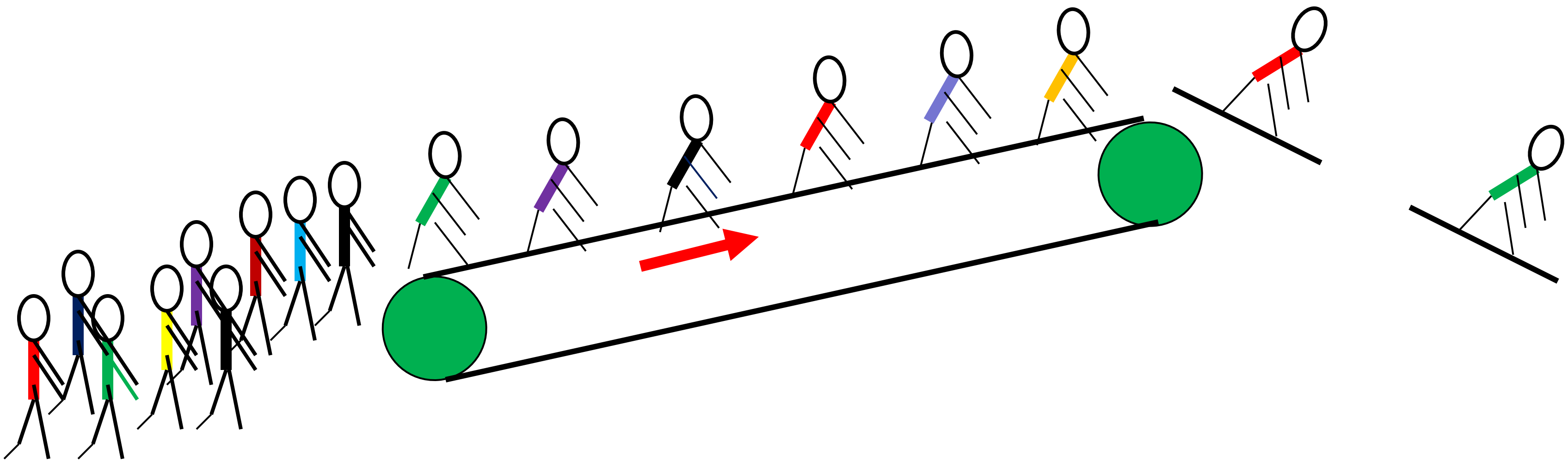
Find the average force required to pull the belt



v : 1.5 m/s

L : 100 m

θ : 20^0



Mass input to the tow at the bottom

$$\left(\frac{dm}{dt}\right)_{in} = 14\,kg/s \qquad v_{rel}^{in} = -1.5\ m/s$$

Mass output from the tow at the top

$$\left(\frac{dm}{dt}\right)_{out} = 14\,kg/s \qquad v_{rel}^{out} = 0$$

No. of skiers on the tow at any time = $\frac{100}{5 \times 1.5} \approx 13$

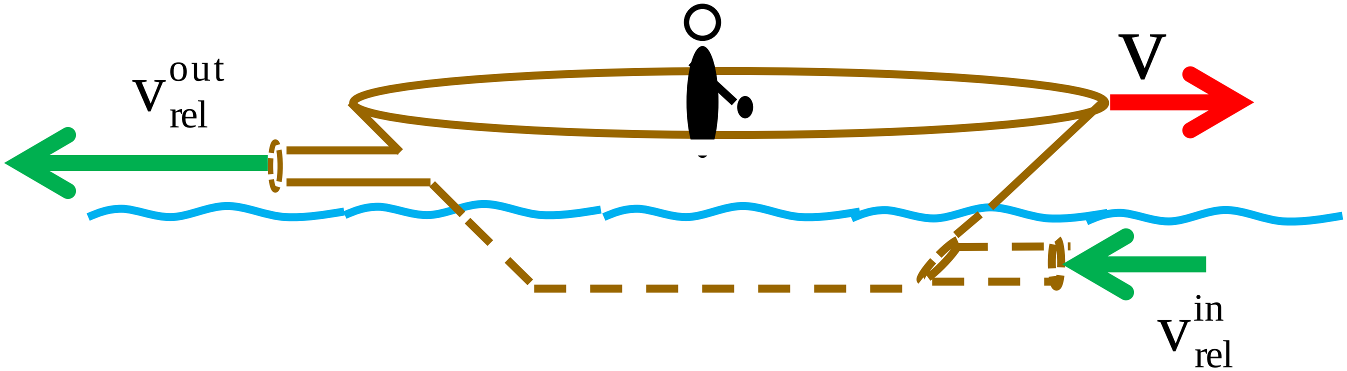
Equation of motion of the tow

$$M\frac{dv}{dt} = F_{pull} - Mg\sin 20^0 + \left(\frac{dm}{dt}\right)_{in} v_{rel}^{in} + \left(\frac{dm}{dt}\right)_{out} v_{rel}^{out}$$

$$\Rightarrow \quad F_{pull} = Mg\sin 20^0 - \left(\frac{dm}{dt}\right)_{in} v_{rel}^{in}$$

$$M = 13 \times 70 = 910 \text{ kg}$$

Example :The Water Jet Boat



$$v_{rel}^{in} = -v \qquad v_{rel}^{out} = -u_0$$

Equation of Motion

$$M\frac{dv}{dt} = -bv + \left(\frac{dm}{dt}\right)_{in} v_{rel}^{in} + \left(\frac{dm}{dt}\right)_{out} v_{rel}^{out}$$

$$= -bv + \gamma(-v) + (-\gamma)(-u_0) = -(b+\gamma)v + \gamma u_0$$

Integration gives

$$v(t) = \frac{\gamma u_0}{b+\gamma}\left[1 - e^{-\frac{b+\gamma}{M}t}\right]$$

$$v_{term} = \frac{\gamma u_0}{b+\gamma}$$