R.R. Mishra

Department Of Physics

BITS Pilani

Mechanics, Oscillations and Waves (MEOW)

Major Division

1. **Mechanics (R.R. Mishra)**

   20 ~ 22 Lectures

2. **Oscillations & Waves (D.D. Pant)**

   20 ~ 22 Lectures

Textbooks :

1. An Introduction to Mechanics : Daniel Kleppner & Robert Kolenkow

2. The Physics of Vibrations & Waves : A. P. French

Mechanics

Chapter No. 2 : Review of Newton's Equations

Chapter No. 3 : Linear Momentum

Chapter No. 4 : Work, Energy & Power

Chapter No. 6 : Angular Momentum

Chapter No. 8 : Non-inertial systems and Fictitious Forces

A world simple enough to be understood, would be too simple to produce a mind that can understand it. **J.D. Barrow**

Chapter 2

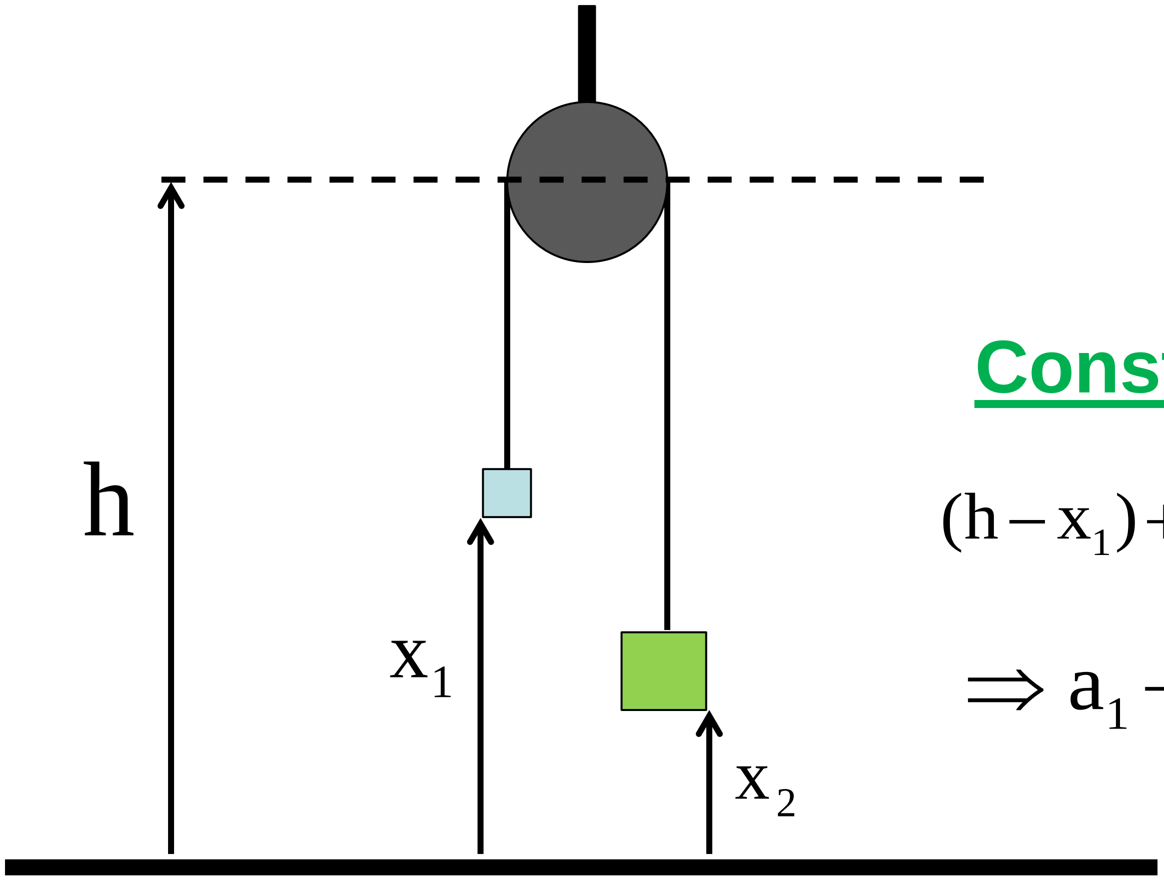
- Constrained Motion

- Newton’s Equations in Polar Coordinates

Constrained Motion

Examples

1. Pulley-mass system

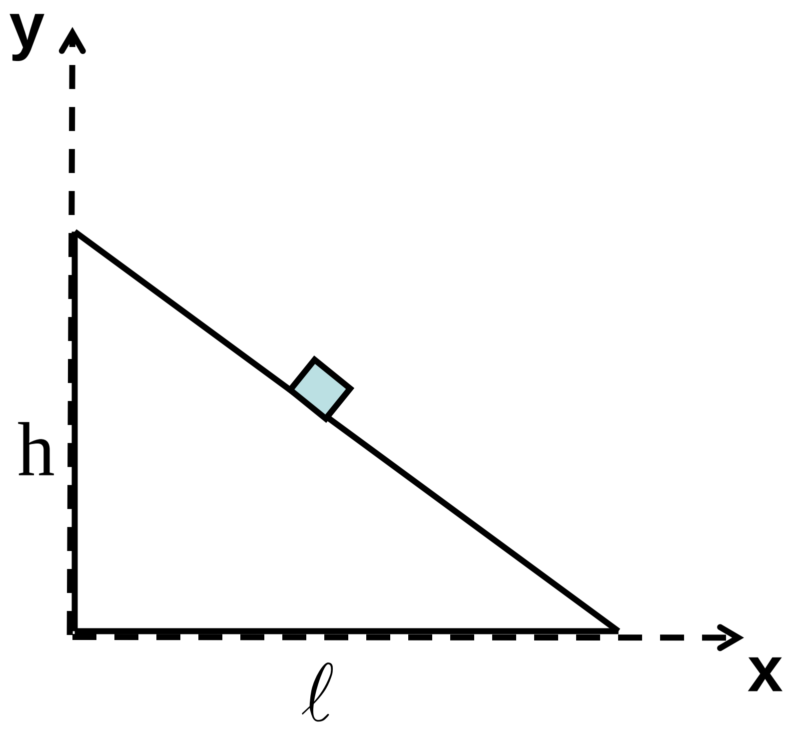


Constraint eq.

$$(h - x_1) + (h - x_2) = \ell$$

$$\Rightarrow a_1 + a_2 = 0$$

2. Block on a fixed wedge

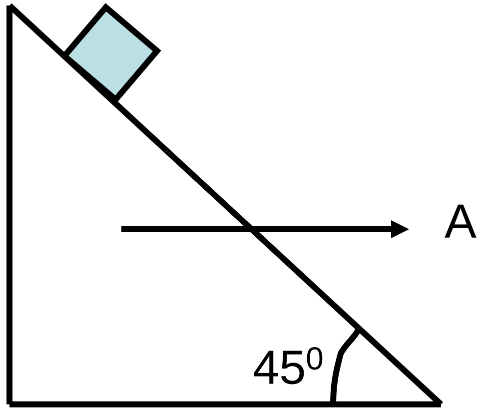


Constraint equation :

$$\frac{x}{\ell} + \frac{y}{h} = 1 \Rightarrow a_y = -\frac{h}{\ell} a_x$$

Prob. 2.16

Block on accelerated wedge

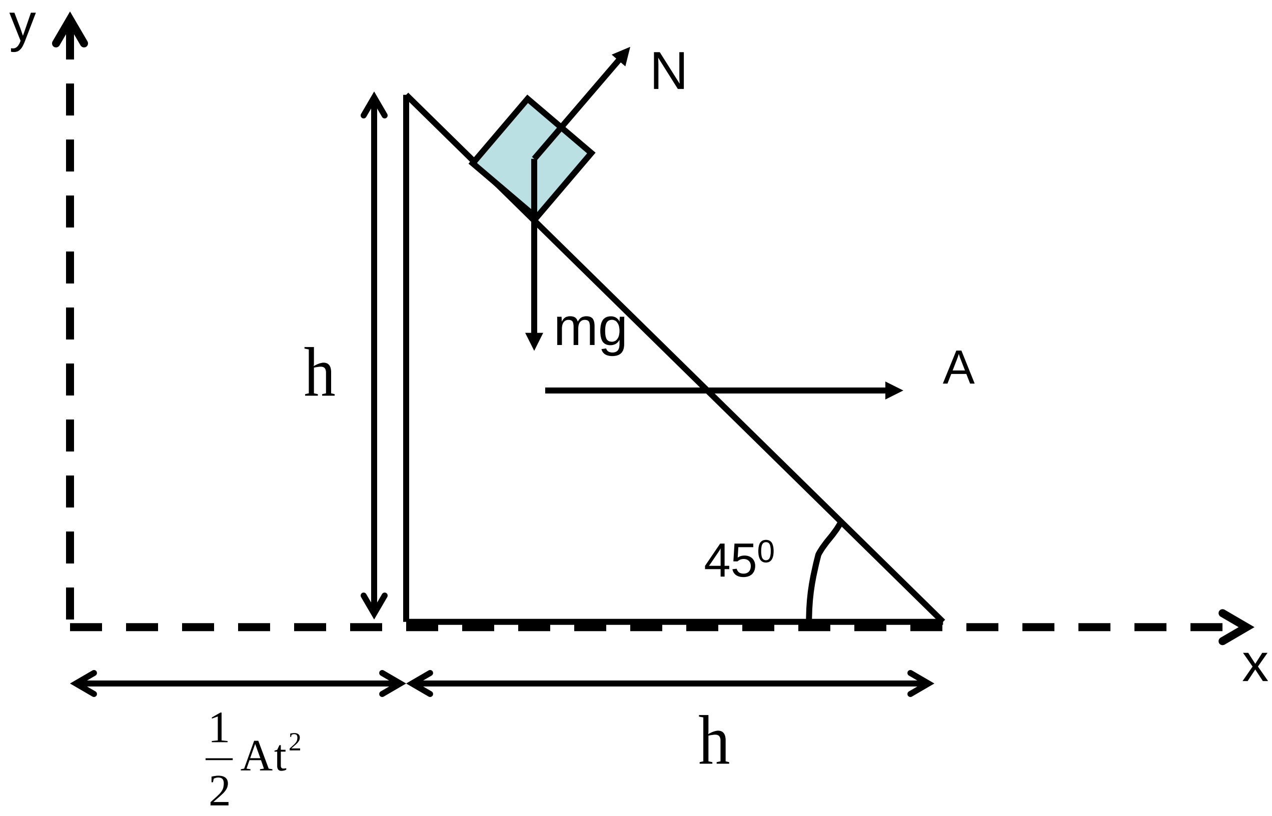


Angle of wedge : 45^0

Acceleration of wedge : A, to right

No friction between block and wedge

Q : What is the acceleration of the block w.r.t. the ground



Constraint Equation :

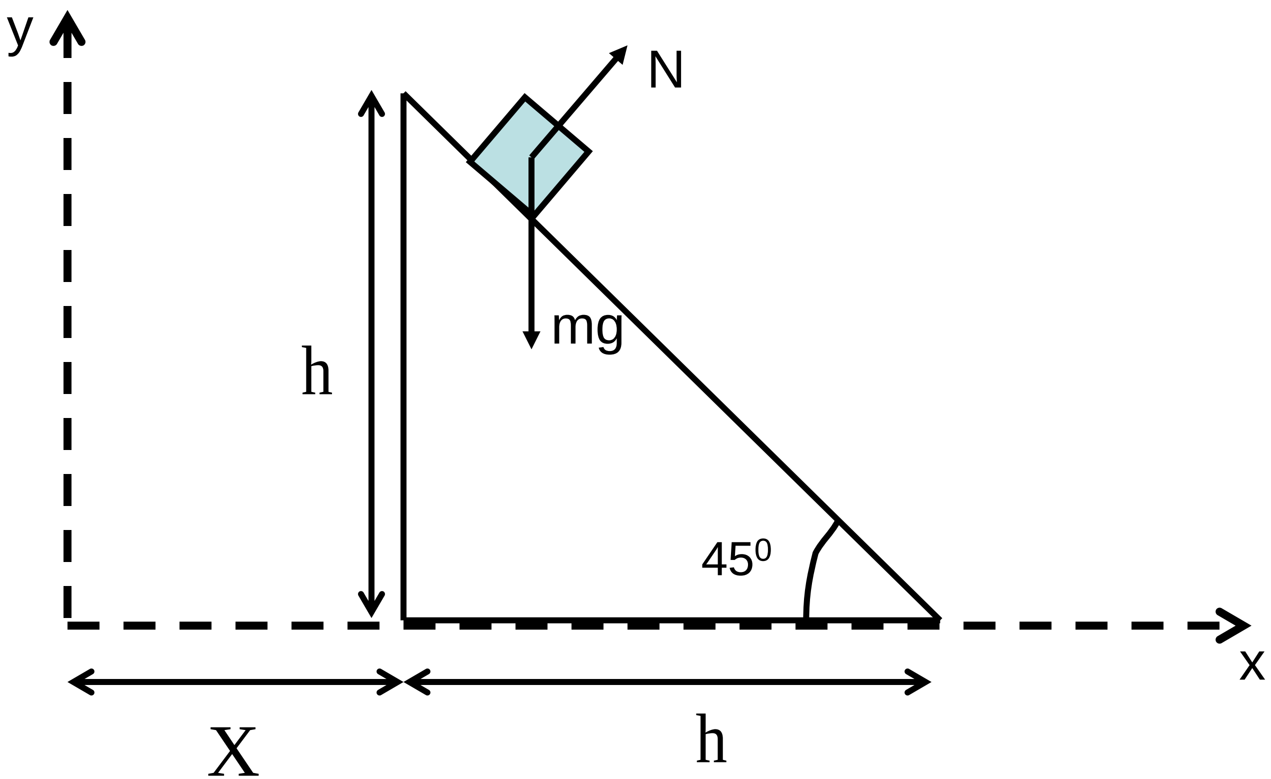
$$x + y = h + \frac{1}{2}At^2 \Rightarrow a_x + a_y = A$$

$$\frac{N}{\sqrt{2}} + \left(\frac{N}{\sqrt{2}} - mg\right) = mA \Rightarrow N = \frac{m}{\sqrt{2}}(A + g)$$

$$\therefore\ a_x = \frac{1}{2}(A+g)\ ;\ a_y = \frac{1}{2}(A-g)$$

The block will climb up the wedge iff A > g

Suppose the wedge is left to itself and is free to move on a frictionless surface.



$$x + y = h + X \Rightarrow a_x + a_y = A_x$$

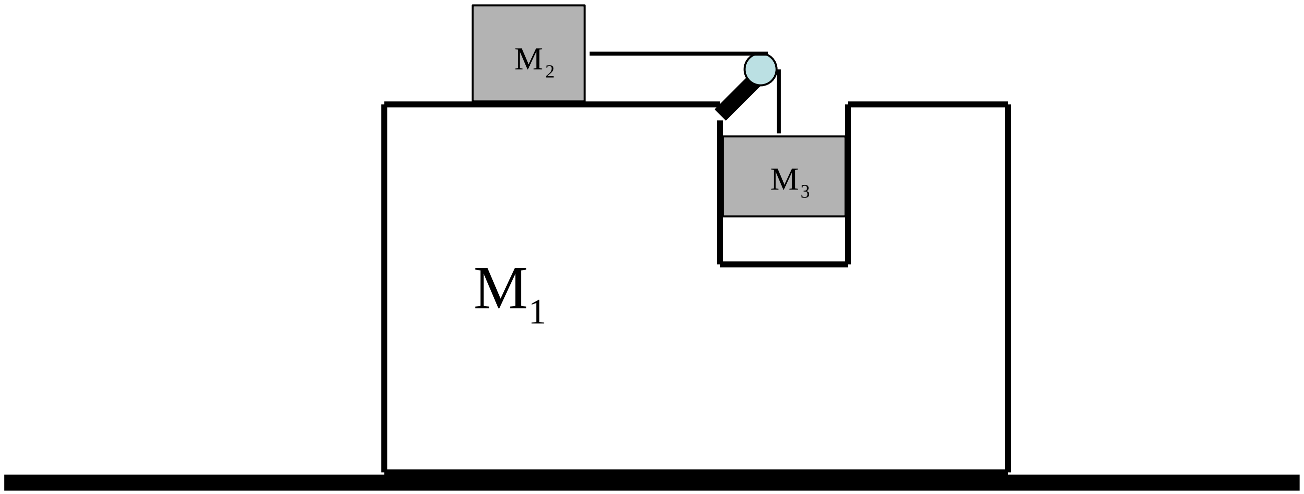
$$\frac{N}{\sqrt{2}} + \left(\frac{N}{\sqrt{2}} - mg\right) = -\frac{N}{\sqrt{2}}$$

(Mass of wedge : m)

$$\Rightarrow \frac{N}{\sqrt{2}} = \frac{mg}{3}$$

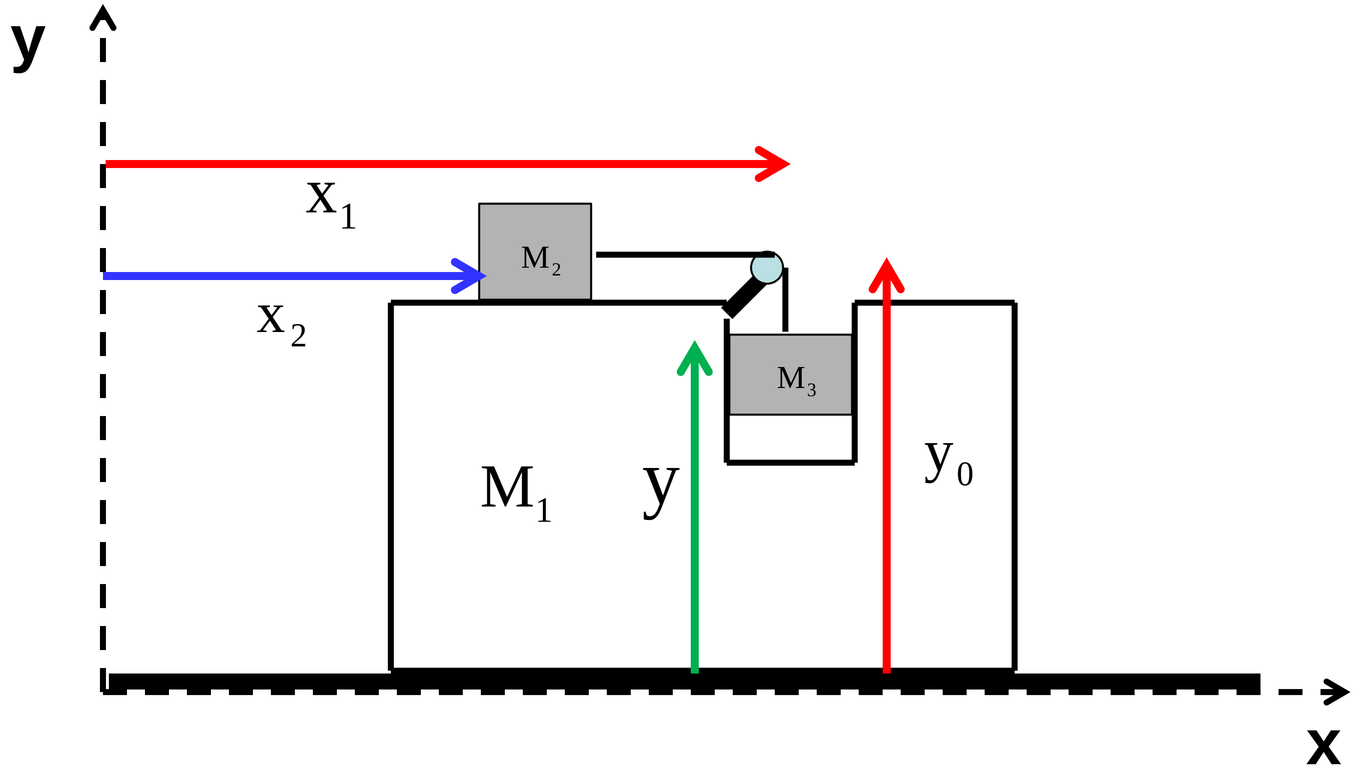
$$\therefore \quad a_x = \frac{g}{3}; \quad a_y = -\frac{2g}{3}; \quad A_x = -\frac{g}{3}$$

Prob. 2.20



The Pedagogic Machine

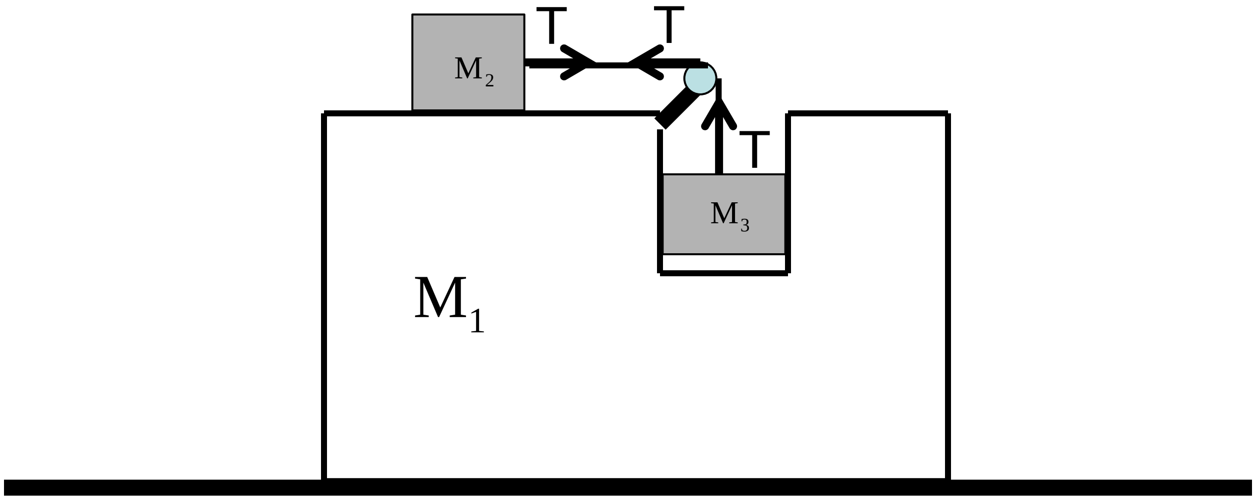
Consider the "pedagogic machine". All surfaces are frictionless. Find the acceleration of block M_1 when the system is released.



Constraint Equation :

$$(x_1 - x_2) + (y_0 - y) = \ell$$

$$\Rightarrow a_1 - a_2 - a_3 = 0$$



$$T = M_2 a_2$$

$$T - M_3 g = M_3 a_3$$

$$a_1 - a_2 - a_3 = 0$$

$$T = -(M_1 + M_3) a_1$$

Four unknowns, T, a_1, a_2 & a_3 **, and four equations !**

All four unknowns can be solved for.

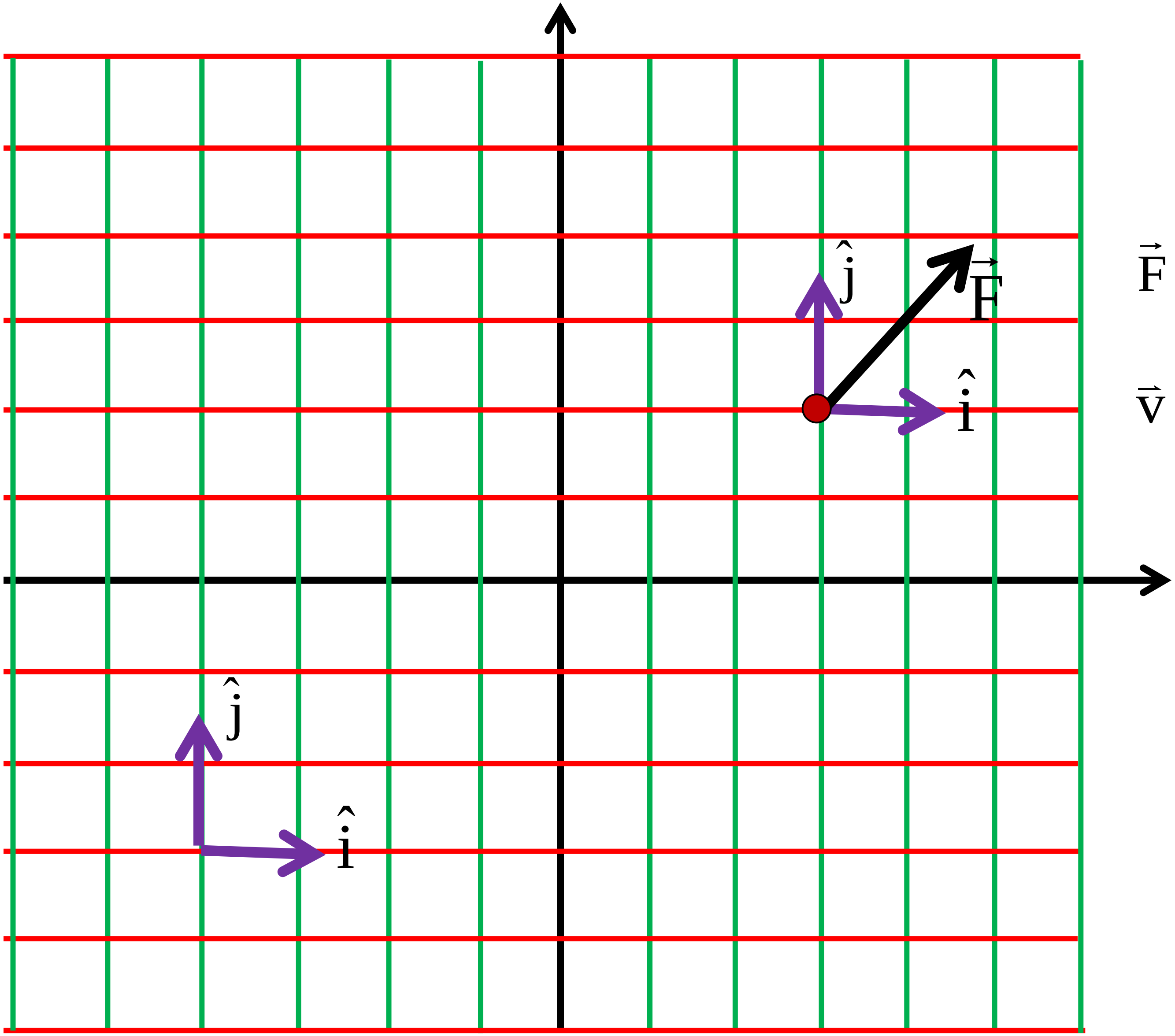
In particular,

$$a_1 = -\frac{M_2 M_3}{M_1 M_2 + M_1 M_3 + 2 M_2 M_3 + M_3^2}\, g$$

Newton's Equations in Polar Co-ordinates

Review of Newton's Eq. in Cartesian Co-ordinates

The Cartesian System



$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j}$$

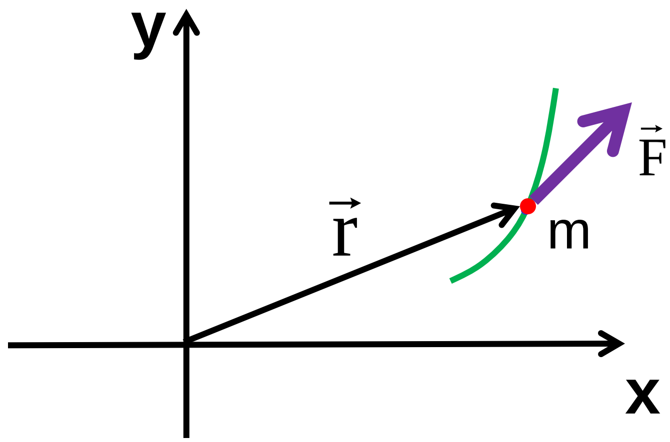
$$\vec{v} = v_x\,\hat{i} + v_y\,\hat{j}$$

Equations of Motion

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

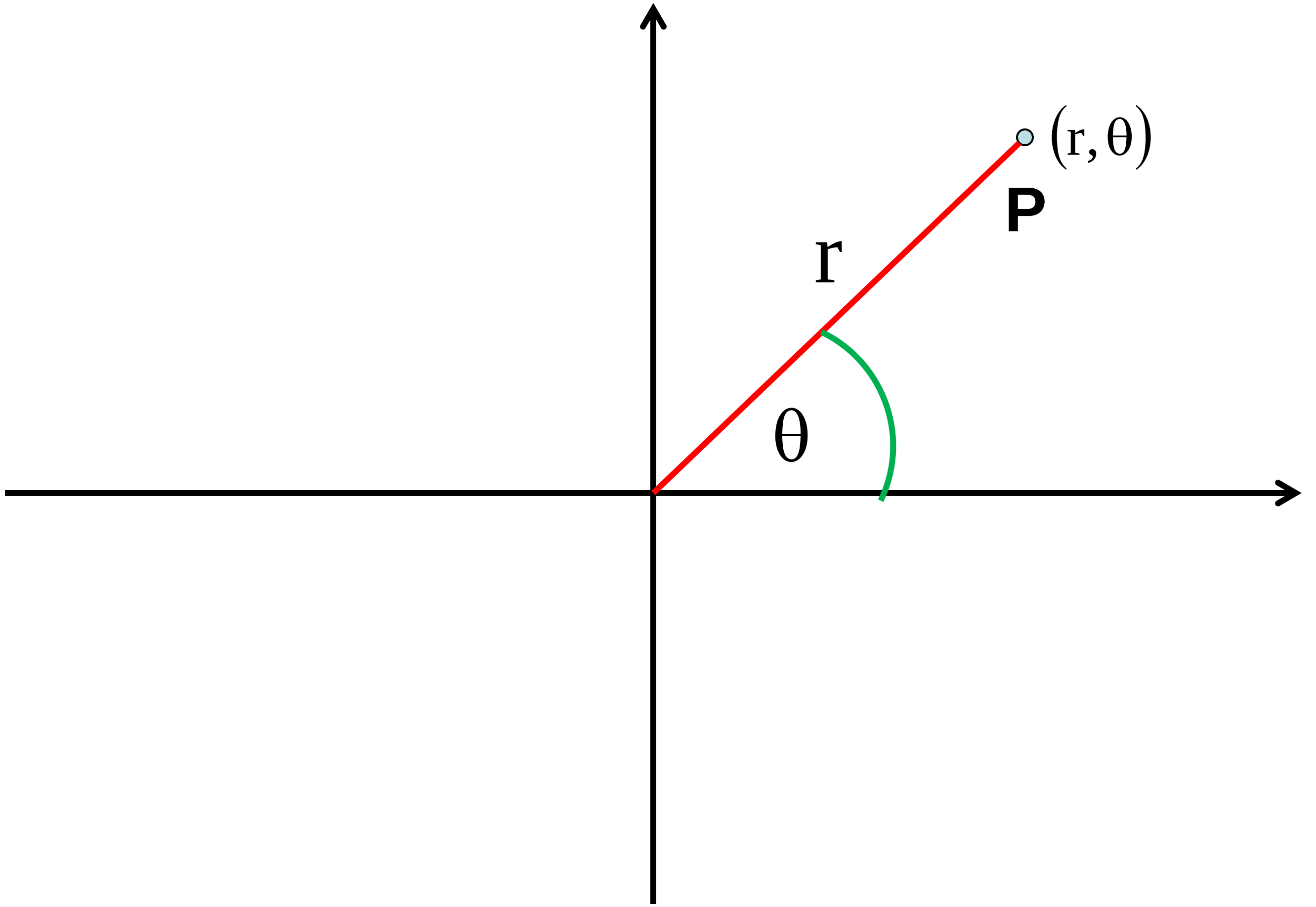
$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}$$



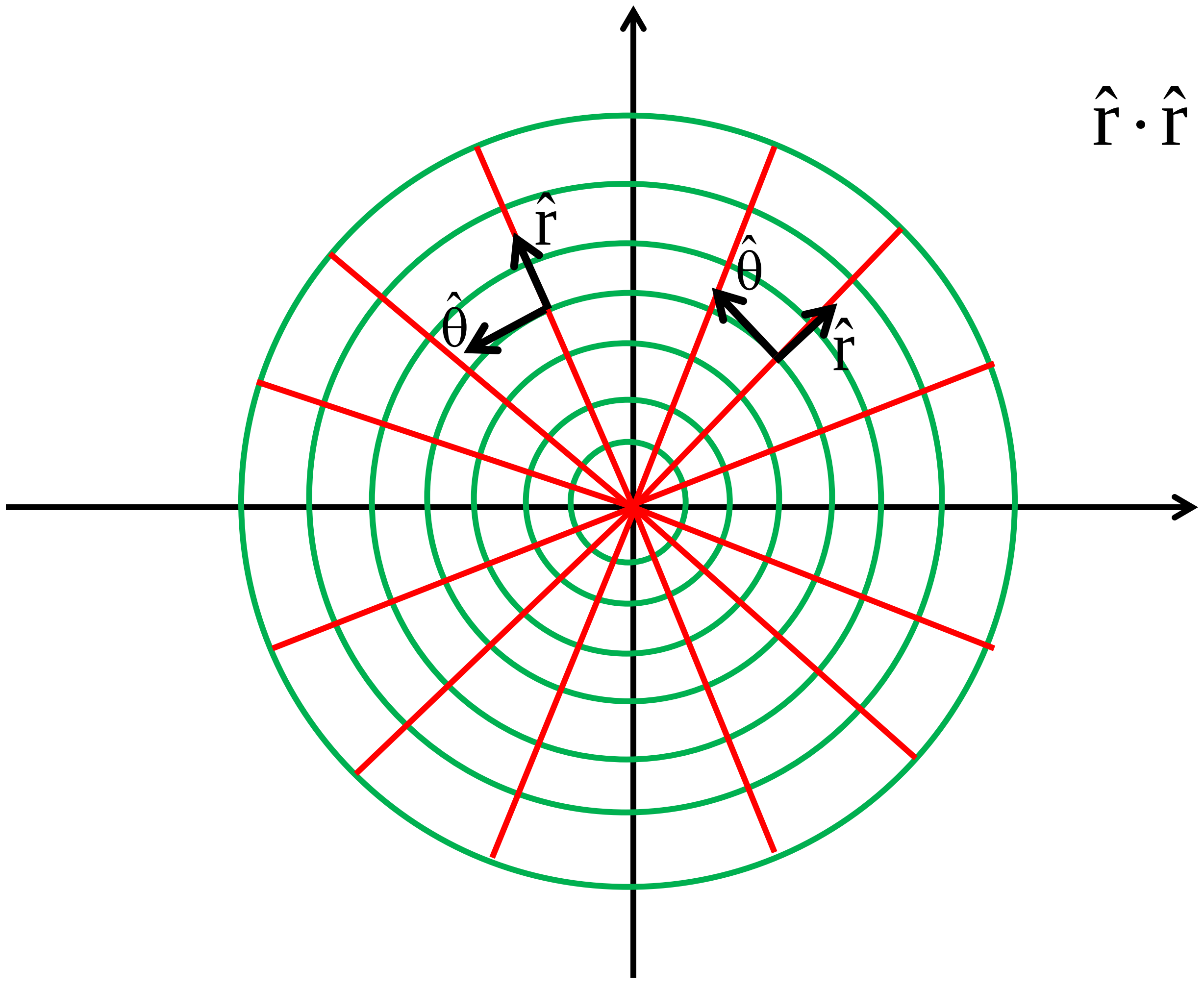
$$m\vec{a} = \vec{F} \;\Rightarrow\; m(\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}) = F_x\,\hat{i} + F_y\,\hat{j}$$

$$\Rightarrow m\ddot{x} = F_x \;\; ; \;\; m\ddot{y} = F_y$$

Polar Co-ordinates



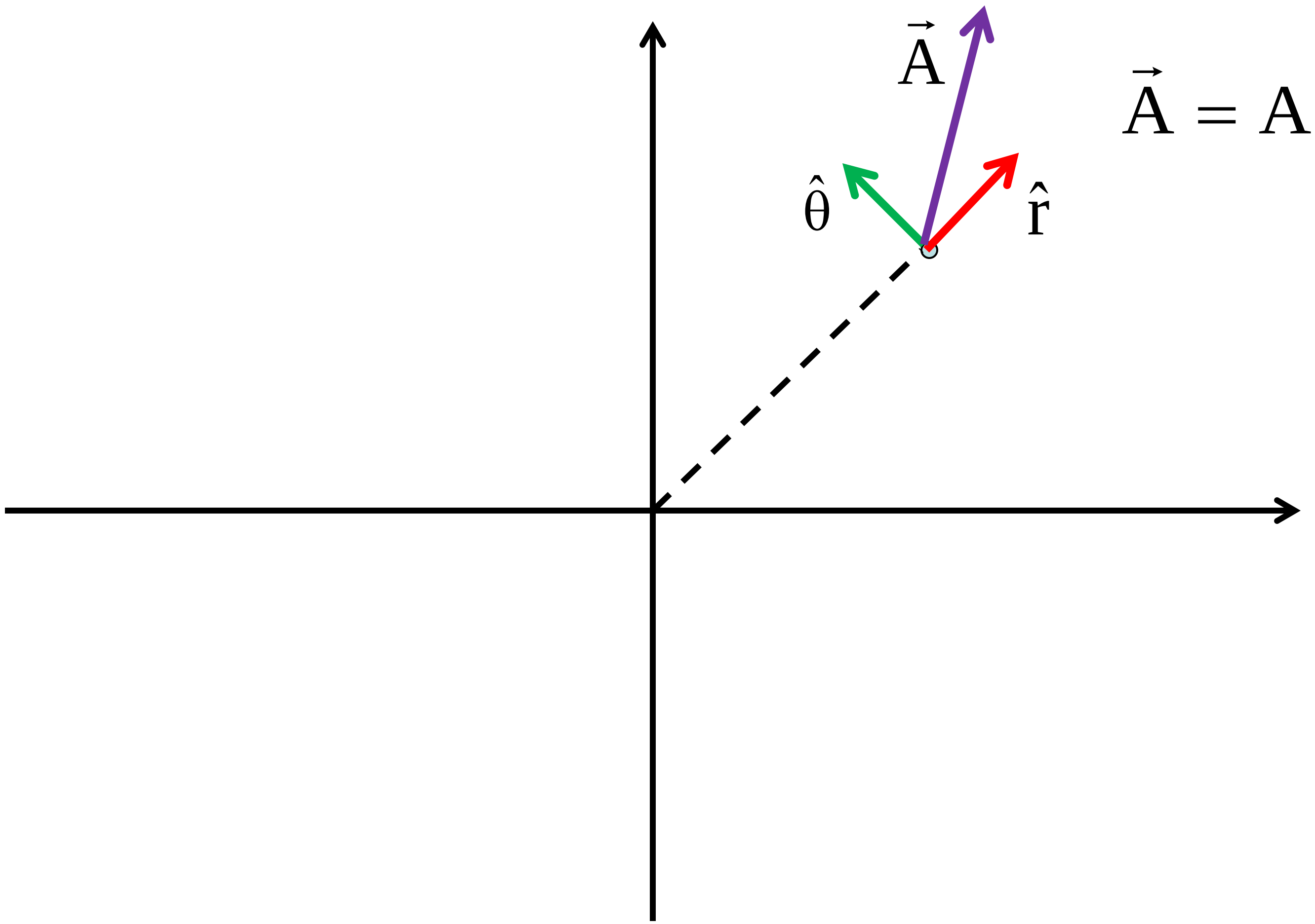
The Polar Grid and Unit Vectors



$$\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1$$

$$\hat{r}\cdot\hat{\theta} = 0$$

Resolving Vectors in Polar Coordinates



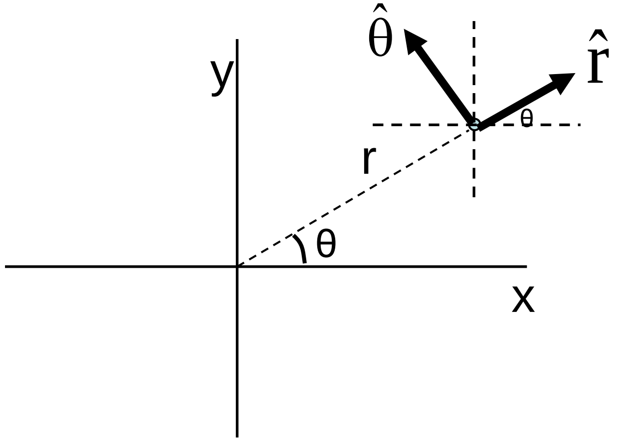
$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta}$$

Unit vectors $(\hat{r}, \hat{\theta})$ in the polar co-ordinates vary from point to point, unlike unit vectors $(\hat{i}, \hat{j})$ in the Cartesian co-ordinates.

Expressing $(\hat{r}, \hat{\theta})$ in terms of $(\hat{i}, \hat{j})$.

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

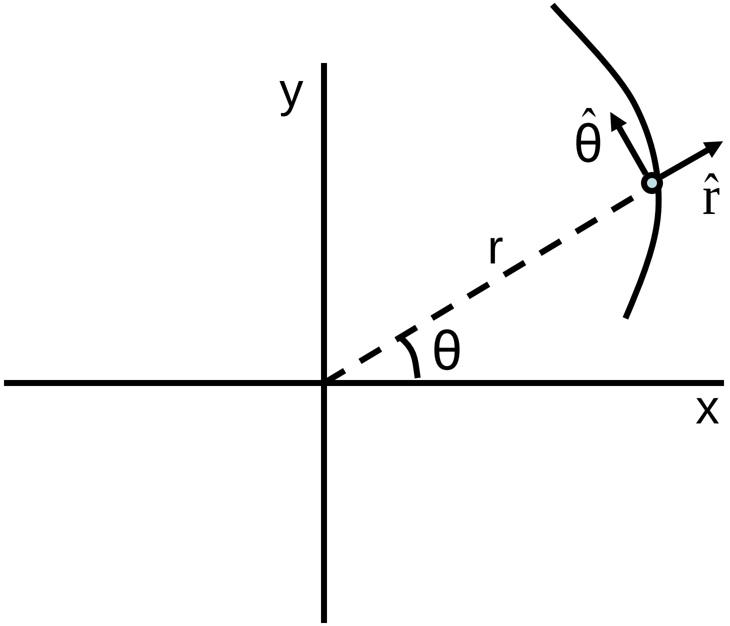


Newton's Equations of Motion in Polar Coordinates

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\dot{\hat{r}} = \dot{\theta}\hat{\theta} \quad \& \quad \dot{\hat{\theta}} = -\dot{\theta}\hat{r}$$



We have,

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

$$= \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} \quad \Rightarrow \; v_r = \dot{r} \;\; \& \;\; v_\theta = r\,\dot{\theta}$$

Another differentiation leads to

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta} + \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} - r\,\dot{\theta}^2\,\hat{r}$$

$$= (\ddot{r} - r\,\dot{\theta}^2)\,\hat{r} + (r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta})\,\hat{\theta}$$

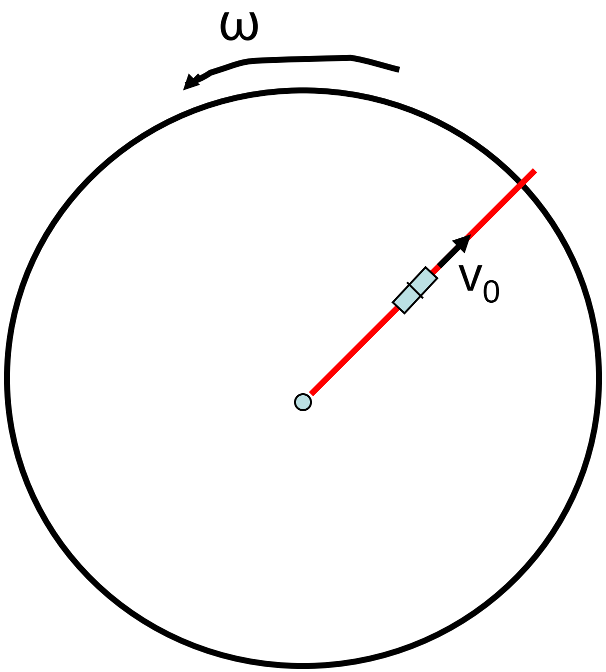
Comparing with $\vec{a} = a_r\,\hat{r} + a_\theta\,\hat{\theta}$

$$a_r = \ddot{r} - r\dot{\theta}^2 \ \& \ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$\left.\begin{aligned} \therefore\ m(\ddot{r} - r\dot{\theta}^2) &= F_r \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= F_\theta \end{aligned}\right\}$$

Newton's Eqs. in polar coordinates

Prob. 2.29



A car moving radially outward on revolving platform.

Ang. Velocity of platform = ω (constant)

Velocity of car w.r.t. platform = v_0

Coeff. of friction = μ

Car starts from centre of platform

Find :

a) Acceleration of car as a function of time using polar coordinates. Show by vector diagram

b) The time at which the car starts to skid

c) Direction of frictional force at the time of skidding

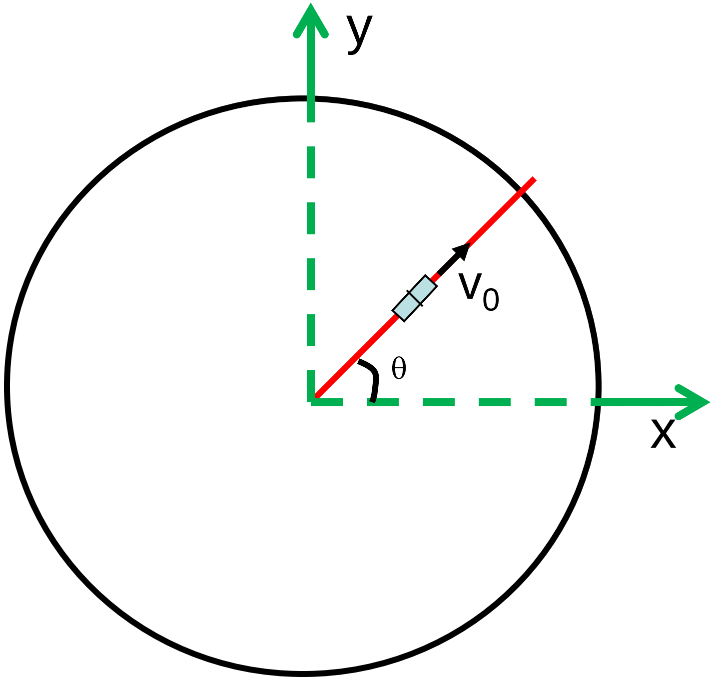
a) We have :

$$r = v_0 t$$

$$\theta = \omega t$$

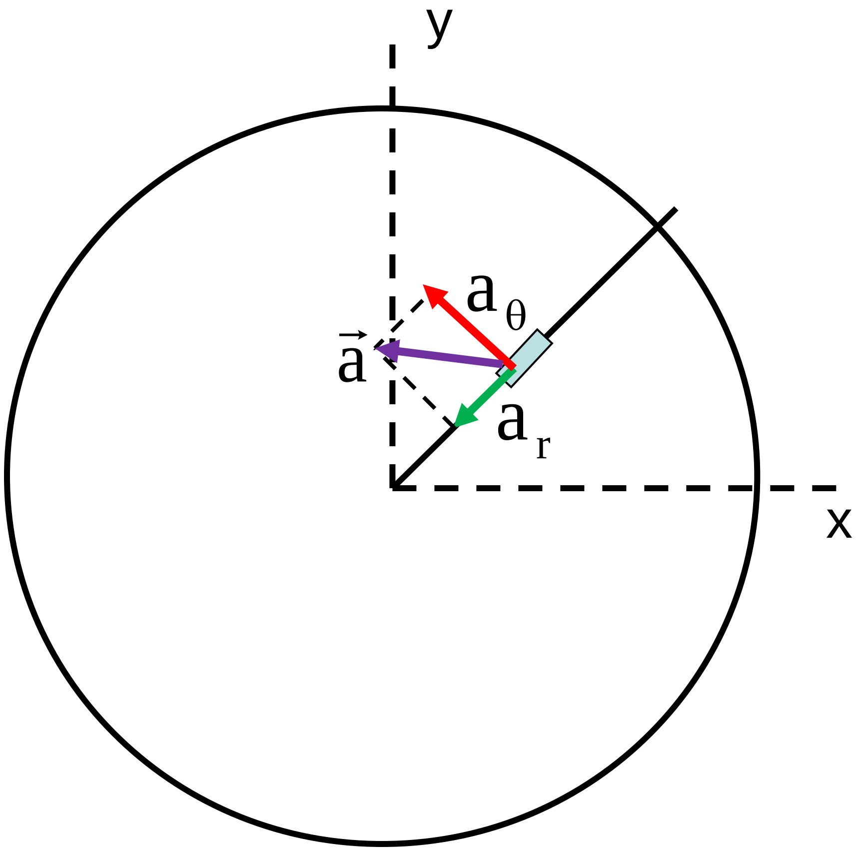
$$\therefore\ a_r = \ddot{r} - r\dot{\theta}^2 = -r\omega^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2v_0\omega$$



$$a_r = -r\omega^2$$

$$a_\theta = 2v_0\omega$$



b) Net force on car :

$$F = \sqrt{F_r^2 + F_\theta^2} \quad = m\sqrt{a_r^2 + a_\theta^2}$$

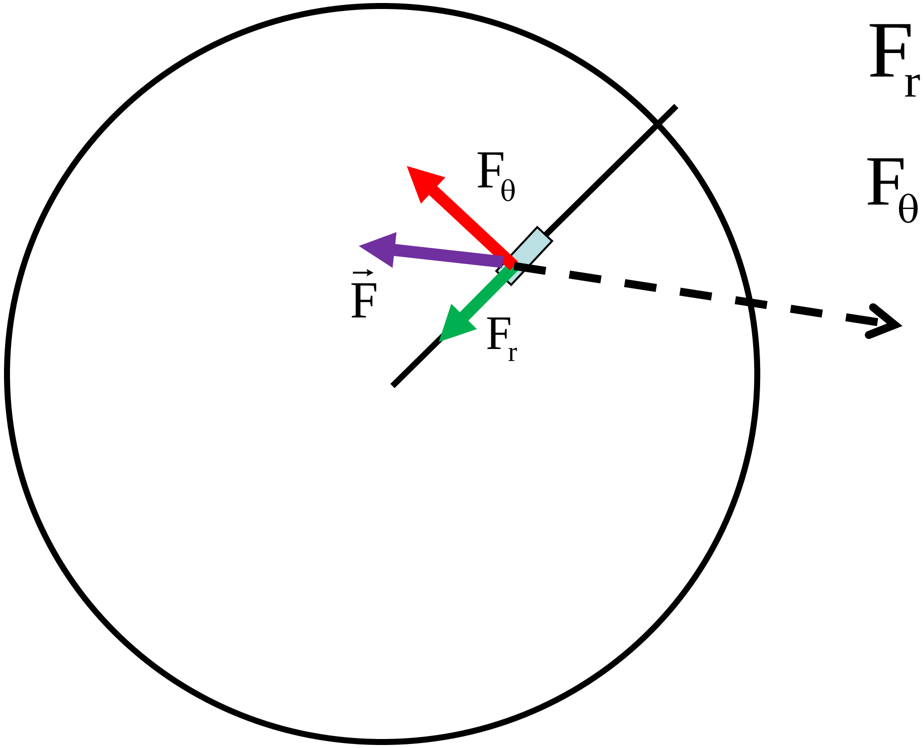
$$= m\sqrt{r^2\omega^4 + 4\omega^2 v_0^2}$$

$$= m\sqrt{v_0^2\omega^4 t^2 + 4\omega^2 v_0^2} = m\omega v_0\sqrt{4 + \omega^2 t^2}$$

The car will start to skid when

$$F = \mu m g \Rightarrow t = \frac{\sqrt{\mu^2 g^2 - 4\omega^2 v_0^2}}{v_0\omega^2}$$

c) Direction of frictional force at the time of skidding



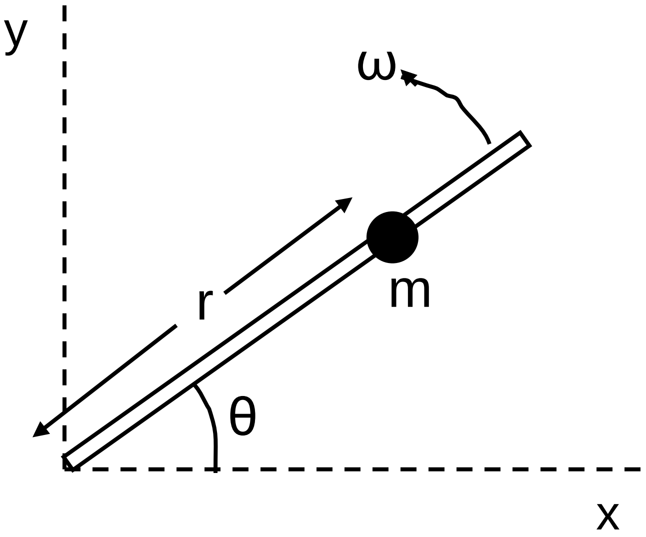
$$F_r = -mr\omega^2$$

$$F_\theta = 2mv_0\omega$$

Direction of skidding

Prob. 2.33

Rod with a mass m on it. Rod rotates with constant angular velocity ωon a horizontal plane and mass free to slide.



i) Show that motion is given by

$$r(t) = Ae^{-\beta t} + Be^{\beta t}$$

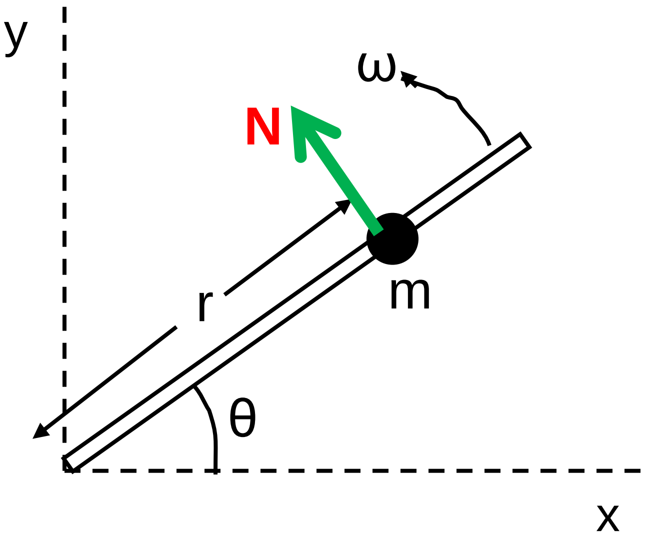
Find β.

ii) Show that for a particular choice of initial conditions, it is possible to obtain a solution such that r continually decreases and that for all other choice, r will eventually increase.

a) The Polar equations are :

$$(\ddot{r} - r\omega^2) = 0$$

$$2m\omega\dot{r} = F_\theta = N$$



The radial equation is

$$\frac{d^2r}{dt^2} = \omega^2 r$$

The coordinate r Should be such a function of t that twice differentiation of r w.r.t. t will be proportional to itself.

An intelligent guess :

$$r(t) = e^{\beta t}$$

Substituting this into the LHS

$$\beta^2 r = \omega^2 r$$

$$\Rightarrow \beta = \pm \omega$$

$\therefore$ Both $e^{\omega t}$ & $e^{-\omega t}$ are solutions for r

The given equation being a linear equation, the most general solution is

$$r(t) = A e^{\omega t} + B e^{-\omega t}$$

where, A & B are constants to be determined

ii) Let the initial conditions be :

$$r(0) = r_0 \;\; \& \;\; \dot{r}(0) = v_0$$

The complete solution is then :

$$r(t) = \frac{1}{2}\left(r_0 + v_0/\omega\right)e^{\omega t} + \frac{1}{2}\left(r_0 - v_0/\omega\right)e^{-\omega t}$$

Now,

$$\frac{dr}{dt} = \frac{1}{2}\left[\left(r_0\omega + v_0\right)e^{\omega t} - \left(r_0\omega - v_0\right)e^{-\omega t}\right]$$

For dr/dt to be negative, the coeff. of $e^{\omega t}$ must be negative.

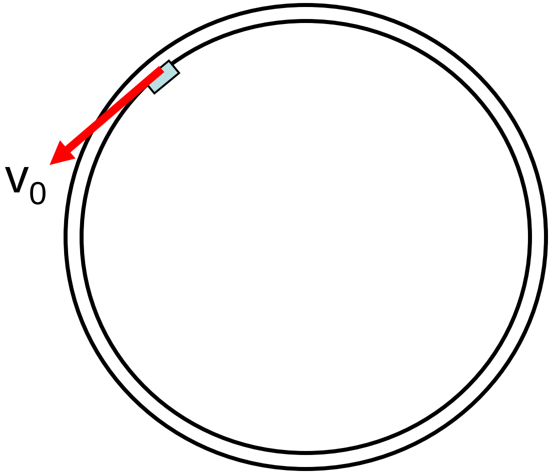
Additional Part

What is the normal force of the rod on the bead?

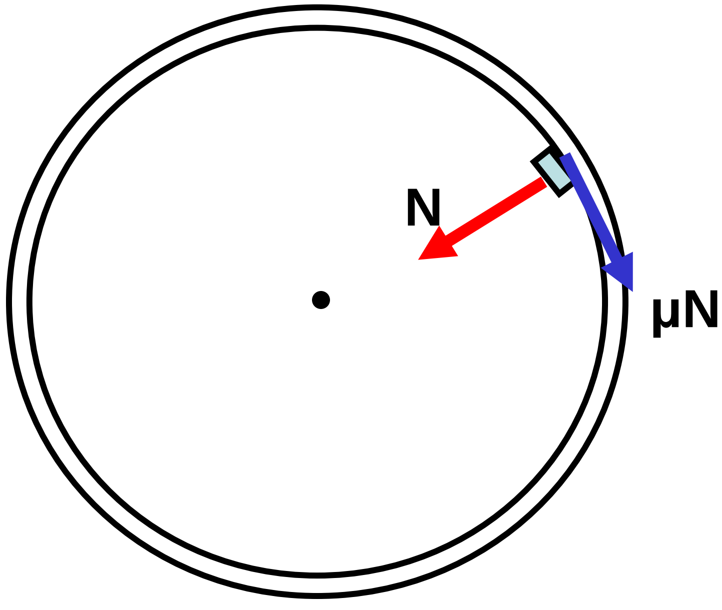
$$N = 2m\omega\dot{r} = m\omega^2\left[\left(r_0 + \frac{v_0}{\omega}\right)e^{\omega t} - \left(r_0 - \frac{v_0}{\omega}\right)e^{-\omega t}\right]$$

Prob. 2.35

A block of mass m slides on the inside of a ring fixed to a frictionless table. It is given an initial velocity of v_0. Coefficient of friction between the ring and the block is μ. Find the velocity and position at a later time t.



ANS :



$$F_r = -N$$

$$F_\theta = -\mu N$$

The polar equations are :

$$m(\ddot{r} - r\dot{\theta}^2) = -mR\dot{\theta}^2 = -N$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mR\ddot{\theta} = -\mu N$$

Substituting N from the first equation in the second

$$\ddot{\theta} = -\mu\dot{\theta}^2$$

Putting $\dot{\theta} = \omega$

$$\frac{d\omega}{dt} = -\mu\omega^2$$

$$\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega'^2} = -\mu\int_0^t dt' \;\Rightarrow\; \left[\frac{1}{\omega} - \frac{1}{\omega_0}\right] = \mu\, t$$

$$\omega = \frac{\omega_0}{1 + \mu\omega_0 t} \qquad \Rightarrow \quad v = \frac{R\, v_0}{R + \mu v_0 t}$$

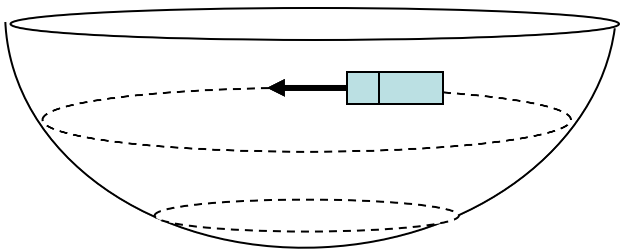
$$\frac{d\theta}{dt} = \frac{\omega_0}{1 + \mu\omega_0 t}$$

$$\int_0^{\theta} d\theta' = \int_0^{t} \frac{dt'}{1 + \mu\omega_0 t'}$$

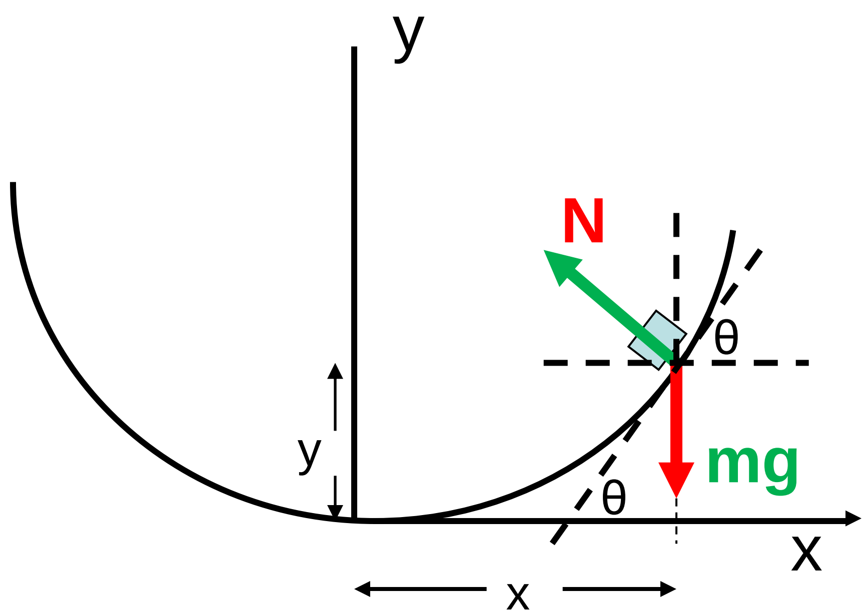
$$\theta = \frac{1}{\mu\omega_0} \ell n \left[1 + \mu\omega_0 t\right]$$

Prob. 2.37

A bowl shaped racing track, on which a racing vehicle can move in horizontal circles without friction.



Q : What should be the equation of the vertical cross section of the track, so that it takes the same time T to circle the track, whatever be its elevation



$$N\cos\theta = mg$$

$$N\sin\theta = \frac{mv^2}{x}$$

$$\therefore \tan\theta = \frac{dy}{dx} = \frac{v^2}{g\,x}$$

$$\text{Since},\ v = \frac{2\pi x}{T},$$

$$\frac{dy}{dx} = \frac{4\pi^2 x}{g\,T^2} = k\,x$$

$$\Rightarrow\ y = k\,\frac{x^2}{2}$$ **(Parabola)**