SVD in finite and Infinite dimensional settings

M. T. Nair

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Plan

- Basic notions:
  1. Orthogonality in $\mathbb{R}^n$.
  2. Projection theorem in $\mathbb{R}^n$
  3. Adjoint of a linear transformation
- Spectral/diagonalization theorem
- SVD in f.d. setting.
- Moore-Penrose inverse in f.d. setting
- Effect of data errors while solving $Ax = b$.
- Compact operators between Hilbert spaces
- Spectral representation of compact operators
- SVD for compact operators
- Generalized solution and Moore-Penrose inverse
- Ill-posedness of compact operator equations
- BHCP
- Ill-posedness of Fredholm integral equations & Applications

SVD in finite dimensional setting

[1] M.T. Nair & A. Singh: *Linear Algebra*, Springer 2018.

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- **Usual inner product** in $\mathbb{R}^d$:

$$\langle x, y\rangle = \sum_{i=1}^{d} x(j)y(j), \quad x, y \in \mathbb{R}^d.$$

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- **Usual norm** on $\mathbb{R}^d$:

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{d} |x(i)|^2}.$$

- For $x, y \in \mathbb{R}^d$, x is **orthogonal to** y if $\langle x, y\rangle = 0$.

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- For $x, y \in \mathbb{R}^d$, x is **orthogonal to** y if $\langle x, y\rangle = 0$.
- $S \subseteq \mathbb{R}^d$ is an **orthonormal set** if for every $x, y \in S$,

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

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- $x = u + v, \quad u \in S, \quad v \in S^\perp$ implies

$$\|x - u\| = \inf_{w \in S} \|x - w\|.$$

Adjoint

Adjoint

Theorem

For every linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$*, there exists a unique linear transformation* $A^* : \mathbb{R}^m \to \mathbb{R}^d$ *satisfying*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall\, x \in \mathbb{R}^n,\ y \in \mathbb{R}^m.$$

Proof.

Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be orthonormal bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we have

$$x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i, \quad y = \sum_{j=1}^{n} \langle y, v_j \rangle v_j.$$

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Then,

$$\begin{aligned} \langle Ax, y \rangle &= \sum_{j=1}^{n} \sum_{i=1}^{n} \langle x, u_i \rangle \langle Au_i, v_j \rangle \langle y, v_j \rangle \\ &= \left\langle x, \sum_{j=1}^{n} \sum_{i=1}^{n} \langle Au_i, v_j \rangle \langle y, v_j \rangle u_i \right\rangle = \langle x, A^* y \rangle, \end{aligned}$$

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$$A^* y := \sum_{j=1}^{n} \sum_{i=1}^{n} \langle Au_i, v_j \rangle \langle y, v_j \rangle u_i, \quad y \in \mathbb{R}^m.$$

Proof continues.

Then, $A^* : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and it satisfies

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \forall\, x \in \mathbb{R}^n,\ y \in \mathbb{R}^m.$$

If $\tilde{A} : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation satisfying

$$\langle Ax, y \rangle = \langle x, \tilde{A} y \rangle \quad \forall\, x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,$$

then we have

$$\langle x, \tilde{A} y \rangle = \langle x, A^* y \rangle \quad \forall\, x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$$

so that $\tilde{A} = A^*$. $\square$

Definition

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then the unique linear transformation $A^* : \mathbb{R}^m \to \mathbb{R}^d$ obtained by Theorem 1 is called the **adjoint** of A. $\diamondsuit$

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A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ which is the adjoint of itself, that is, $A^* = A$, is called a **self-adjoint operator**. $\diamondsuit$

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- Linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint iff

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Spectral/diagonalization theorem

Spectral/diagonalization theorem

Theorem

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ *be a self-adjoint operator. Then there exists real numbers* $\lambda_1, \ldots, \lambda_n$ *and an orthonormal basis* $\{u_1, \ldots, u_n\}$ *such that*

$$Ax = \sum_{i=1}^{n} \lambda_i \langle x, u_i \rangle u_i, \quad x \in \mathbb{R}^n.$$

Proof involves the following steps

- A has an eigenvalue.
- Every eigenvalue is real.
- Take all eigenvalues $\lambda_1, \ldots, \lambda_k$ and consider

$$S = N(A - \lambda_1 I) \oplus \cdots \oplus N(A - \lambda_k).$$

- Show that $S^\perp = \{0\}$.
- $\mathbb{R}^n = S$
- Take orthonormal basis $\{u_1^{(i)}, \ldots, u_n^{(i)}\}$ of of $N(A - \lambda_i I)$, and

$$\{u_1, \ldots, u_n\} = \bigcup_{i=1}^{k} \{u_1^{(i)}, \ldots, u_n^{(i)}\}.$$

- If $[A]$ is the matrix representation of A with respect to the orthonormal basis $\{u_1, \ldots, u_n\}$, then writing $u_1, \ldots, u_n$ as column vectors, we obtain

$$[A] = U^T D U,$$

where

$$U = [u_1, \ldots, u_n], \quad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Singular value decomposition (SVD)

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$A^*A : \mathbb{R}^n \to \mathbb{R}^n$$

is a self-adjoint operator. Indeed, for every $x, y \in \mathbb{R}^n$,

$$\langle x, A^*Ay\rangle = \langle Ax, Ay\rangle = \langle Ay, Ax\rangle = \langle y, A^*Ax\rangle = \langle A^*Ax, y\rangle.$$

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Hence

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Therefore, by spectral theorem, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and an orthonormal basis $\{u_1, \ldots, u_n\}$ for $\mathbb{R}^n$ such that

$$A^*Ax = \sum_{i=1}^{n} \lambda_i \langle x, u_i\rangle u_i, \quad x \in \mathbb{R}^n.$$

In this case, $A^*Au_j = \lambda_j u_j$ so that

$$\lambda_j = \langle u_j, \lambda_j u_j \rangle = \langle u_j, A^*Au_j \rangle = \langle Au_j, Au_j \rangle \geq 0$$

and

$$\lambda_j = 0 \iff Au_j = 0.$$

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Hence, we assume, without loss of generality,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Let

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Now, let

$$\sigma_j = \sqrt{\lambda_j}, \quad v_j = \frac{Au_j}{\sigma_j}, \quad j = 1, \ldots, k.$$

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Since $x = \sum_{i=1}^{n} \langle x, u_i \rangle u_i$, we have

$$Ax = \sum_{i=1}^{n} \langle x, u_i \rangle Au_i.$$

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- $\{v_1, \ldots, v_k\}$ is an orthonormal set:

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$$Ax = \sum_{i=1}^{k} \langle x, u_i \rangle Au_i = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i$$

Thus, we have proved the following theorems:

Theorem

(SVD) *If* $A : \mathbb{R}^n \to \mathbb{R}^m$ *is a non-zero linear transformation, then there exist orthonormal sets* $\{u_1, \ldots, u_k\}$ *and* $\{v_1, \ldots, v_k\}$ *in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and real numbers* $\sigma_1, \ldots, \sigma_k$ *such that* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ *and*

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

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$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

Theorem

(SVD) *Let $A \in \mathbb{R}^{m \times n}$ be non-zero matrix. Then there exist $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{m \times k}$ with orthonormal column n-vectors and m-vectors, respectively, and real numbers $\sigma_1, \ldots, \sigma_k$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ such that*

$$A = U^T \Sigma V, \quad \Sigma := diag(\sigma_1, \ldots, \sigma_k).$$

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with SVD

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

Moore-Penrose inverse

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with SVD

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

Then we have

$$R(A) = \mathrm{span}\{v_1, \ldots, v_k\}, \quad N(A)^\perp = \mathrm{span}\{u_1, \ldots, u_k\}.$$

For $y \in \mathbb{R}^m$, let

$$x^\dagger := \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\sigma_i} u_i.$$

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Then we have

$$Ax^\dagger := \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\sigma_i} Au_i = \sum_{i=1}^{k} \langle y, v_i \rangle v_i = Py,$$

where $P : \mathbb{R}^m \to \mathbb{R}^m$ is the orthogonal projection onto $R(A)$.

Hence,

$$\|Ax^\dagger - y\| = \|Py - y\| = \inf_{x\in\mathbb{R}^n} \|Ax - y\|.$$

Thus,

- $x^\dagger$ is a least-square solution of the equation

$$Ax = y.$$

- Among all the least-square solutions, $x^\dagger$ has the least norm, and it is the only one having the least norm.

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- Among all the least-square solutions, $x^\dagger$ has the least norm, and it is the only one having the least norm.
  This follows from the fact that $x^\dagger \in N(A)^\perp$ and using projection theorem.

Definition

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with SVD

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

Definition

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with SVD

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

Then the linear transformation $A^\dagger : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$A^\dagger y = x^\dagger := \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\sigma_i} u_i$$

is called the **Moore-Penrose inverse** of A. ◇

We observe that,

$$A^\dagger Ax = \sum_{i=1}^k \frac{\langle Ax, v_i\rangle}{\sigma_i} u_i = \sum_{i=1}^k \frac{\langle x, A^* v_i\rangle}{\sigma_i} u_i = \sum_{i=1}^k \langle x, u_i\rangle u_i = Qx,$$

so that

$$A^\dagger A = Q,$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $N(A)^\perp$.

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where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $N(A)^\perp$. In particular, if A in injective, then $n \leq m$ and $k = n$ so that

$$A^\dagger A = I_{\mathbb{R}^n}.$$

We observe that,

$$A^\dagger A x = \sum_{i=1}^{k} \frac{\langle Ax, v_i\rangle}{\sigma_i} u_i = \sum_{i=1}^{k} \frac{\langle x, A^* v_i\rangle}{\sigma_i} u_i = \sum_{i=1}^{k} \langle x, u_i\rangle u_i = Qx,$$

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where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $N(A)^\perp$.
In particular, if A in injective, then $n \le m$ and $k = n$ so that

$$A^\dagger A = I_{\mathbb{R}^n}.$$

- If $n = m$ and A is injective, then it is bijective and $A^\dagger = A^{-1}$.

Effect of data erros

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with SVD

$$Ax = \sum_{i=1}^{k} \sigma_i \langle x, u_i \rangle v_i, \quad x \in \mathbb{R}^n.$$

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Then we have

$$A^\dagger y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\sigma_i} u_i, \quad y \in \mathbb{R}^m.$$

Thus, for $y, \tilde{y} \in \mathbb{R}^m$, we have

$$A^\dagger y - A^\dagger \tilde{y} = \sum_{i=1}^{k} \frac{\langle y - \tilde{y}, v_i \rangle}{\sigma_i} u_i$$

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so that

$$\|A^\dagger y - A^\dagger \tilde{y}\|^2 = \sum_{i=1}^{k} \frac{|\langle y - \tilde{y}, v_i \rangle|^2}{\sigma_i^2} \geq \frac{|\langle y - \tilde{y}, v_k \rangle|^2}{\sigma_k^2}.$$

In particular, if

$$\tilde{y} = y + \sqrt{\sigma}_k v_k$$

then we have

$$\|\tilde{y} - y\| = \sqrt{\sigma}_k \quad \text{but} \quad \|A^\dagger y - A^\dagger \tilde{y}\| = \frac{1}{\sqrt{\sigma}_k}.$$

- *A small error in the data can produce large error in the solution.*

An illustration

Example

For $\alpha \neq 0$, let

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

Then

$$\|Ax - b\|^2 = |\alpha x_1 - b_1|^2 + |b_2|^2.$$

Hence, $\inf \|Ax - b\|$ is attained at $x = \begin{bmatrix} b_1/\alpha \\ x_2 \end{bmatrix}$ and for this x, $\|x\|^2 = |b_1/\alpha|^2 + |x_2|^2$ is least for $x_2 = 0$. Hence,

$$x^\dagger = \begin{bmatrix} b_1/\alpha \\ 0 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} 1/\alpha & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\dagger A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

- $\|x^\dagger - \tilde{x}^\dagger\| = |b_1 - \tilde{b}_1|/\alpha$ can be large for small $|b_1 - \tilde{b}_1|$.

SVD in infinite dimensional setting

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.

Let $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ be orthonormal sets in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let (σ_n) be a sequence of positive real numbers such that

$$\sigma_n \to 0 \quad \text{and} \quad \sigma_1 \geq \sigma_2 \geq \cdots .$$

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$$\sigma_n \to 0 \quad \text{and} \quad \sigma_1 \geq \sigma_2 \geq \cdots .$$

Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by

$$Tx = \sum_{j=1}^{\infty} \sigma_j \langle x, u_j \rangle v_j, \quad x \in \mathcal{H}_1. \qquad (*)$$

- T is a bounded linear operator, since

$$\sum_{j=1}^{\infty} \sigma_j^2 |\langle x, u_j \rangle|^2 \leq \sigma_1^2 \|x\|^2.$$

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$$\|T\| = \sigma_1.$$

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Since $\sigma_n \to 0$, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\sigma_{n_k} < 1/k$. Let

$$y = \sum_{j=1}^{\infty} \sigma_{n_j} v_{n_j}.$$

Since $\sum_{j=1}^{\infty} |\sigma_{n_j}|^2 < \infty$, $y \in \mathcal{H}_2$.

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Since $\sum_{j=1}^{\infty} |\sigma_{n_j}|^2 < \infty$, $y \in \mathcal{H}_2$.

Now, for $x \in \mathcal{H}_1$, if $Tx = y$, then we must have

$$\langle Tx, v_{n_j} \rangle = \langle y, v_{n_j} \rangle, \quad \forall j \in \mathbb{N},$$

i.e., $\sigma_{n_j} \langle x, u_{n_j} \rangle = \sigma_{n_j}$ for all $j \in \mathbb{N}$

$$\Rightarrow \quad \langle x, u_{n_j} \rangle = 1 \quad \forall j \in \mathbb{N},$$

which is not possible, since $\langle x, u_n \rangle \to 0$.

- Thus, the equation

$$Tx = y$$

need not have a solution.

[2] M.T. Nair: M.T. Nair: Functional Analysis: A First Course, PHI-Learning 2002, Second Edition 2021 (Chapter 14)

- Thus, the equation
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need not have a solution.
- Let $y \in R(T)$ and $Tx = y$. For $n \in \mathbb{N}$, let
$$y_n := y + \sqrt{\sigma}_n v_n, \quad x_n = x + u_n/\sqrt{\sigma}_n.$$
Then $Tx_n = y_n$. Note that

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$$\|y_n - y\| = \sqrt{\sigma}_n \to 0 \quad \text{but} \quad \|x_n - x\| = \frac{1}{\sqrt{\sigma}_n} \to \infty.$$

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Thus:

- The problem of solving the equation $Tx = y$ is **ill-posed**[2] .

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Hence, $\|T - T_n\| \leq \sigma_{n+1} \to 0$.

- (T_n) is a norm approximation of T.
- T is a *compact operator*.

Definition

A bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called a **compact operator** if closure of $\{Tx : \|x\| \le 1\}$ is compact in $\mathcal{H}_2$. $\diamondsuit$

[3] M.T. Nair, *Functional Analysis: A First Course*, Second Edition PHI Learning, New Delhi, 2021.

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The answer is in the affirmative, thanks to *spectral theorem for a compact self-adjoint operators*[3], [4] .

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Adjoint of bounded operators

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As a consequence of *Riesz representation theorem*, we have:

Theorem

For every bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$*, there exists a unique bounded linear operator* $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ *such that*

$$\langle Tx, y \rangle_{\mathcal{H}_2} = \langle x, T^* y \rangle_{\mathcal{H}_1} \quad \forall (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2.$$

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The operator T^* defined in the last theorem is called the **adjoint** of T.

A bounded operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ called a **self-adjoint operator** if $A^* = A$, that is,

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, x, y \in \mathcal{H}. \qquad \diamondsuit$$

Theorem

(Spectral theorem) *If* $A : \mathcal{H} \to \mathcal{H}$ *is a compact self-adjoint operator of infinite rank, then there exist an orthonormal set* $\{u_1, u_2, \ldots\}$ *in* $\mathcal{H}$ *and a sequence* (λ_n) *of real numbers with* $\lambda_n \to 0$ *such that*

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j \quad \forall\, x \in \mathcal{H}.$$

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Then $A := T^*T$ is a compact, positive, self-adjoint operator on $\mathcal{H}_1$. Let

$$T^*Tx = \sum_{j=1}^{\infty} \lambda_j \langle x, u_j \rangle u_j \quad \forall\, x \in \mathcal{H}_1$$

be the spectral representation of T^*T.

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$$\langle v_i, v_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle Tu_i, Tu_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle u_i, T^*Tu_j \rangle = \frac{\sigma_j^2}{\sigma_i \sigma_j} \langle u_i, u_j \rangle = \delta_{ij}.$$

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Thus,

- $\{v_j : j \in \mathbb{N}\}$ is an orthonormal basis of $R(T)$.

Note that, for $x \in \mathcal{H}_1$,

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Thus, we have proved the following theorem.

Theorem

(SVD) *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a compact operator of infinite rank, then*

$$Tx = \sum_{j=1}^{\infty} \sigma_j \langle x, u_j \rangle v_j \quad \forall\, x \in \mathcal{H}_1,$$

where $\{u_1, u_2, \ldots\}$ and $\{v_1, v_2, \ldots\}$ are orthonormal bases of $N(T)^\perp$ and $R(T)$, respectively and (σ_n) is a sequence of positive real numbers with $\sigma_n \to 0$.

[5] called *Piccard condition*

Generalized solution and Moore-Penrose inverse

Let us assume, without loss of generality, that

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- For $y \in \mathcal{H}_2$,

$$x^\dagger := \sum_{j=1}^{\infty} \frac{\langle y, v_j \rangle}{\sigma_j} u_j$$

is well-defined iff[5]

$$\sum_{j=1}^{\infty} \frac{|\langle y, v_j \rangle|^2}{\sigma_j^2} < \infty.$$

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$$D := \left\{ y \in \mathcal{H}_2 : \sum_{j=1}^{\infty} \frac{|\langle y, v_j \rangle|^2}{\sigma_j^2} < \infty \right\}.$$

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For $y \in D$, we have

[6] since $\{v_n : n \in \mathbb{N}\}$ is an onb of $\overline{R(T)}$.

For $y \in D$, we have

$$Tx^\dagger = \sum_{j=1}^{\infty} \langle y, v_j \rangle u_j = Py,$$

where $P : \mathcal{H}_2 \to \mathcal{H}_2$ is the orthogonal projection[6] onto $\overline{R(T)}$.

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Hence,

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- Among all the least-square solutions, $x^\dagger$ has the least norm, and it is the only one having the least norm.

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- Among all the least-square solutions, $x^\dagger$ has the least norm, and it is the only one having the least norm.

This is the consequence of the fact that $x^\dagger \in N(T)^\perp$.

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Definition

The map $T^\dagger : D \to \mathcal{H}_1$ defined by

$$T^\dagger y := \sum_{j=1}^{\infty} \frac{\langle y, v_j \rangle}{\sigma_j} u_j, \quad y \in D$$

is called the **Moore-Penrose invere** of T. ♢

[7] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, Second Edition (Under preparation), World Scientific.

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Remark: Morre-Penrose inverse can also be defined for a general bounded operator between Hilbert spaces[7]. In such case, it can be shown that $T^\dagger$ is a bounded operator iff $R(T)$ is closed. In the case under discussion, we already having an operator with non-closed range.

[7] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, Second Edition (Under preparation), World Scientific.

Ill-Posedness of $Tx = y$

Now, if $y, \tilde{y} \in D$, then

$$T^\dagger(y - \tilde{y}) = \sum_{i=1}^{\infty} \frac{\langle y - \tilde{y}, v_i \rangle}{\sigma_i} u_i.$$

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BHCP

Consider the heat equation

$$\frac{\partial u}{\partial t}(s,t) = c^2 \frac{\partial^2 u}{\partial s^2}(s,t), \quad 0 < s < \ell, t > 0. \tag{1}$$

If $f_0 \in L^2[0,\ell]$, and

$$u(\cdot, t) := \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \langle f_0, \varphi_n \rangle \varphi_n, \tag{2}$$

where

$$\lambda_n := \frac{n\pi c}{\ell}, \quad \varphi_n(s) := \sqrt{\frac{2}{\ell}} \sin(\lambda_n s),$$

then $u(\cdot,\cdot)$ satisfies (1) with

$$u(s,0) = f(s) \quad \text{a.e.}.$$

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- The answer is in the affirmative iff

$$\sum_{n=1} e^{2\lambda_n^2(\tau-t)}|\langle g,\varphi_n\rangle|^2 < \infty. \qquad (3)$$

The function g has to be too smooth!

Proof.

From (2),

$$\langle g, \varphi_n \rangle = \langle u(\cdot, \tau), \varphi_n \rangle = e^{-\lambda_n^2 \tau} \langle f_0, \varphi_n \rangle$$

so that

$$\langle f_0, \varphi_n \rangle = e^{\lambda_n^2 \tau} \langle g, \varphi_n \rangle$$

so that

$$u(\cdot, t) := \sum_{n=1}^{\infty} e^{\lambda_n^2 (\tau - t)} \langle g, \varphi_n \rangle \varphi_n. \tag{4}$$

$\Rightarrow$

$$\sum_{n=1}^{\infty} e^{2\lambda_n^2 (\tau - t)} |\langle g, \varphi_n \rangle|^2 < \infty.$$

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The relations (3) and (4) implies

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- K_t is a compact, positive, self adjoint operator with *exponentially decaying* eigenvalues (singular values)

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- If $k(\cdot,\cdot)$ is non-degenerate, then the problem of solving $(*)$ is ill-posed.

Ill-Posed integral equations appear in many applications.

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Ill-Posed Integral Equations

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For example:

- Computerized tomography, [Radon transform]
- Geological prospecting, [Abel integral equations]
- Inverse heat conduction problems, [with smooth kernel]

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$x(\cdot)$ is the density of mineral deposit;
$y(\cdot)$: is the gravimetric measurements

- Inverse heat conduction problems: BHCP

$$\int_0^\ell k(s,\xi) f_0(\xi) d\xi = g(s),$$

$$k(s,\xi) := \sum_{n=1}^{\infty} e^{-\lambda_n^2(\tau - t_0)} \sin \frac{n\pi s}{\ell} \sin \frac{n\pi \xi}{\ell}, \quad \lambda_n = \frac{n\pi c}{\ell}.$$

$g := u(\cdot, \tau)$: temperature at time τ;
$f_0 := u(\cdot, t_0)$: temperature at time t_0.

[8] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, World Scientific 2009 (Second Edition - Under preparation)

How to deal with ill-posed and ill-conditioned problems?

Regularization theory[8] is the answer!

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- There are many books on inverse and ill-posed problems.

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- There are many books on inverse and ill-posed problems.
- The journals
  - Inverse Problems,
  - Journal of Inverse and Ill-Posed Problems,
  - Inverse Problems in Science and Engineering,
  - Inverse Problems and Imaging

  are some of the journals exclusively devoted to this area.

[8] M.T. Nair, *Linear Operator Equations: Approximation and Regularization*, World Scientific 2009 (Second Edition - Under preparation)

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Thank you all
for your attention!

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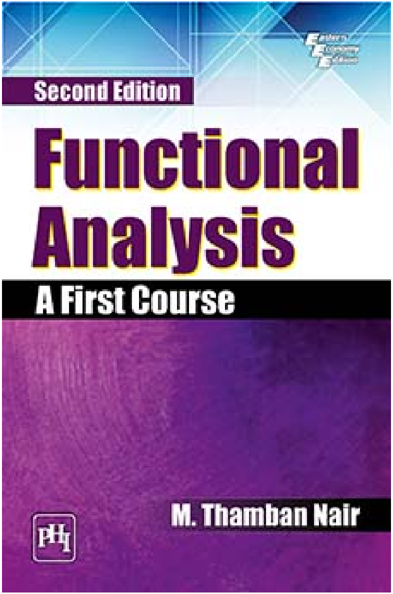

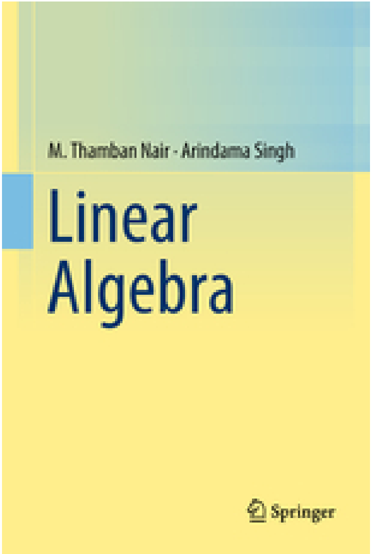
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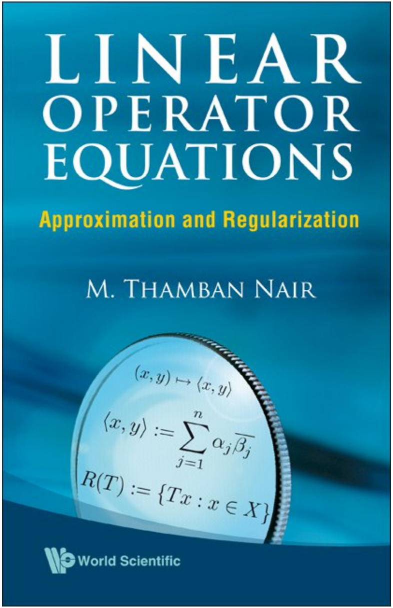
LINEAR
OPERATOR
EQUATIONS
Approximation and Regularization
M. THAMBAN NAIR
$(x, y) \mapsto \langle x, y\rangle$
$\langle x, y\rangle := \sum_{j=1}^{n} \alpha_j \overline{\beta_j}$
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World Scientific

Measure and Integration, CRC Press, 2019

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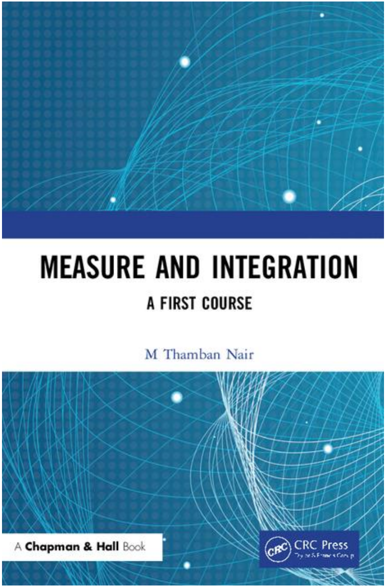

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